The Precise Definition of a Limit

Section 2.4

Precise definition of limit

Assume f is defined on an interval containing a, except not necessarily at a.

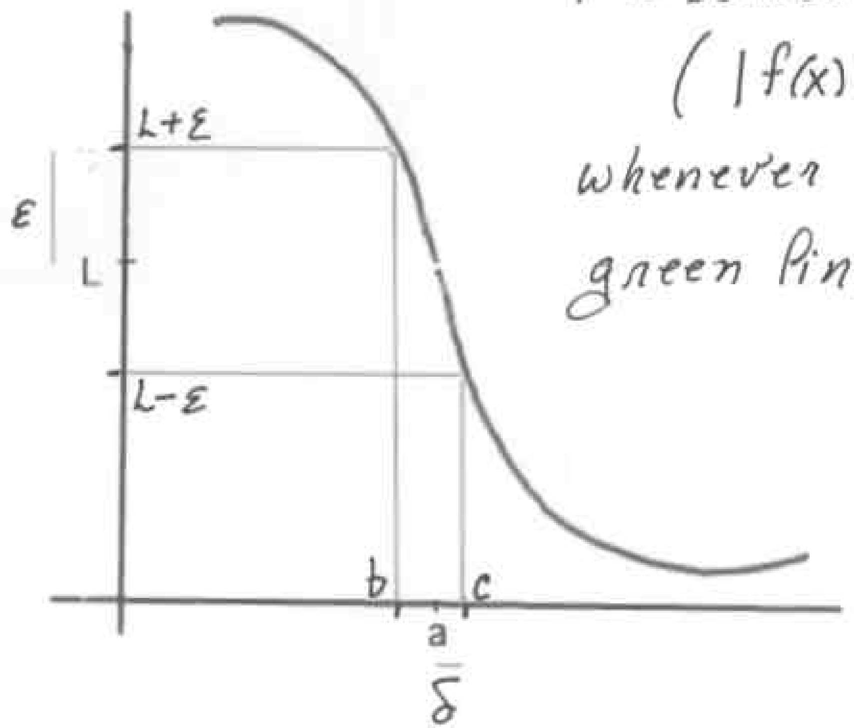
$\lim_{x \to a} f(x) = L$ means:
for every number $\varepsilon > 0$ we can find $\delta > 0$ such that

$$|f(x) - L| < \varepsilon \text{ whenever } 0 < |x-a| < \delta.$$

The idea behind the definition is: we can make $f(x)$ as close to L as we may wish (within distance ε) **solely** by making x close to a.

This definition tells us what we must do to **prove** that $\lim_{x \to a} f(x) = L$ — we must tell how to find δ for any ε.

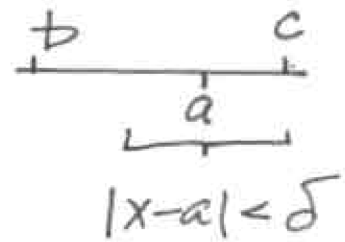
ε–δ Limit Picture



f is between the red lines

$\left(|f(x) - L| < \varepsilon \right)$

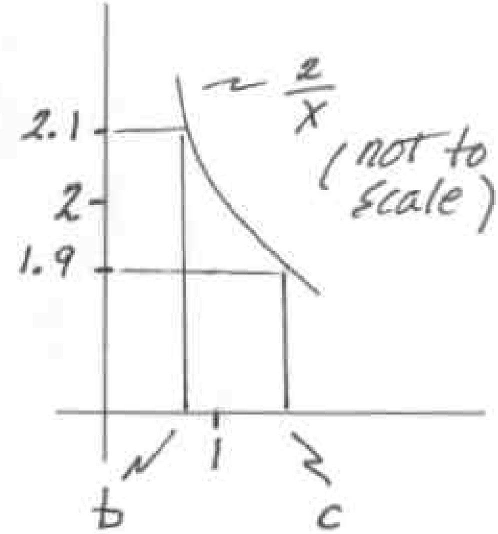
whenever x is between the green lines.



δ will be the minimum of $|b-a|$ & $|c-a|$

Example
(finding a δ for just one ε)

Find δ when $f(x) = \frac{2}{x}$, $a = 1$ and $\varepsilon = 0.1$



We want $\left| \frac{2}{x} - 2 \right| < 0.1$

whenever $|x-1| < \delta$

- find b & c; see which is closer to 1

$$\frac{2}{b} = 2.1, \text{ so } b = \frac{2}{2.1} = \frac{20}{21} = 1 - 1/21$$

$$\frac{2}{c} = 1.9, \text{ so } c = \frac{2}{1.9} = \frac{20}{19} = 1 + 1/19$$

$|b-1| = 1/21$ & $|c-1| = 1/19$

smaller, so Take $\delta = 1/21$

Example (a Limit Proof)

Prove, with an ε–δ argument, that $\lim_{x \to 2} 3x + 1 = 7$.

Given any $\varepsilon > 0$, we must find $\delta > 0$ so that

$$|3x+1-7| < \varepsilon \quad \text{whenever} \quad |x-2| < \delta \qquad *$$

$$\begin{aligned} |3x+1-7| < \varepsilon \quad & \text{when } |3x-6| < \varepsilon \\ & \text{when } 3|x-2| < \varepsilon \\ & \text{when } |x-2| < \varepsilon/3 \end{aligned}$$

Conclusion: For $\delta = \varepsilon/3$, $*$ is True.

(Proofs are harder for non-linear f)

(The limit laws of 2.3 are proved with ε-δ proofs.)