Continuity

Section 2.5

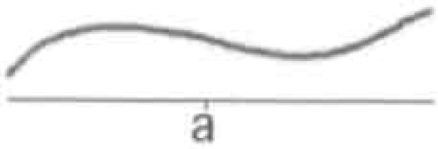
Definition of Continuity

A function f is continuous at a means $\lim_{x \to a} f(x) = f(a)$.

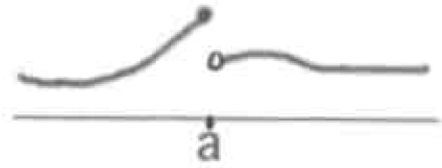
This definition requires 3 things:

1. f must be defined at a
2. $\lim_{x \to a} f(x)$ must exist
3. $\lim_{x \to a} f(x)$ must equal $f(a)$

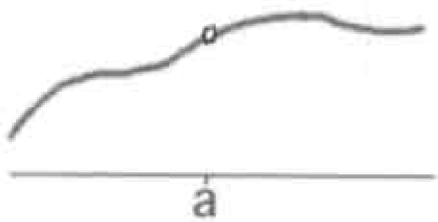
Picture Examples of Continuity and Discontinuity



continuous at a



not continuous at a
jumps not allowed

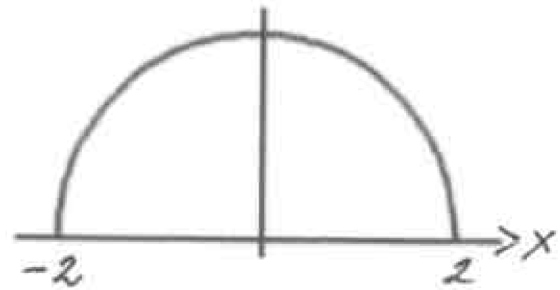


not continuous at a – f not defined at a

but – we could define f(a) to make f continuous at a — a called a removable discontinuity.

Continuity on an Interval

$f(x) = \sqrt{4 - x^2}$



f only defined on $[-2, 2]$

f is continuous from the left

at 2: $\lim\limits_{x \to 2^-} f(x) = f(2)$

f is continuous from the right at -2:

$$\lim_{x \to -2^-} f(x) = f(-2)$$

We say this f is continuous on the interval $[-2, 2]$

Building Continuous Functions

- The limit laws imply that any sum, difference, or product of continuous functions is continuous. Also any quotient, provided the denominator $\neq 0$ at a.

- Since constant functions as well as $f(x) = x$ are continuous everywhere, we see that polynomials $p(x)$ are continuous everywhere, and rational functions $p(x)/q(x)$ are continuous at any point a where $q(a) \neq 0$.

- If g is continuous at a, and if f is continuous at $g(a)$, then the composition $f(g(x))$ is continuous at a (i.e. the composition of continuous functions is continuous).

- power functions, trig and inverse trig functions, exponentials and logarithms are continuous wherever they are defined.

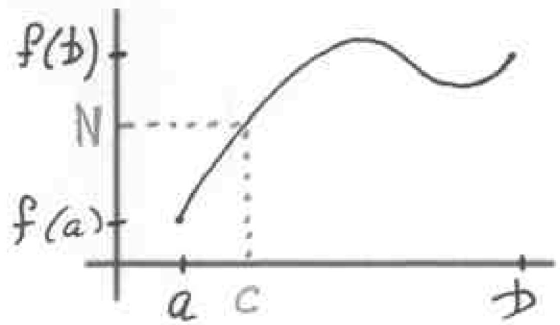
Examples of Continuous Functions

- $\sin x$ Trig function, defined everywhere
  so: continuous everywhere
- $\sin(x^2+1)$ Trig (poly), defined everywhere
  so: continuous everywhere
- $\tan x = \frac{\sin x}{\cos x}$ Trig — not defined where $\cos x = 0$
  so: cont. except at odd multiples of $\pi/2$
- $\frac{\ln x}{x^2-1}$ log/poly, ln only defined for $x>0$
  denom. vanishes at ± 1.

  so: continuous for $0<x<1$ & for $x>1$

Intermediate Value Theorem

Assume f is continuous at each point of a closed interval $[a, b]$. If N is a number between $f(a)$ and $f(b)$, then there exists at least one number c in (a, b) satisfying $f(c) = N$.



- We say a continuous f assumes every intermediate value between $f(a)$ and $f(b)$.

Example Using Intermediate Value Th.

Show that $x^3 + x + 1 = 0$ must have a solution in [-1, 0].

We have a continuous $f = x^3 + x + 1$ on $[-1, 0]$.

$f(-1) = -1$ $\quad\quad\quad f(0) = +1$

Since 0 lies between -1 & $+1$, the Int. Value Th. says there must be a # c in $(-1, 0)$ satisfying $c^3 + c + 1 = 0$.

- A graphing calculator is, in effect, programmed to assume functions continuous, whether or not they are.