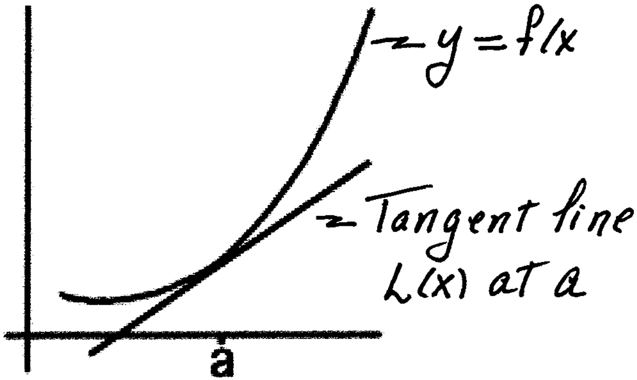
Linear Approximation and Differentials

Section 3.11

The Linear Approximation



When x is near a, $L(x)$ is an approximation to $f(x)$.

$L(x)$ contains $(a, f(a))$ and has slope $f'(a)$.

$L(x) = f(a) + f'(a)(x-a)$ is called the <u>linearization of f at a</u>.

$f(x) \approx f(a) + f'(a)(x-a)$ is called the <u>linear approximation of f at a</u>.

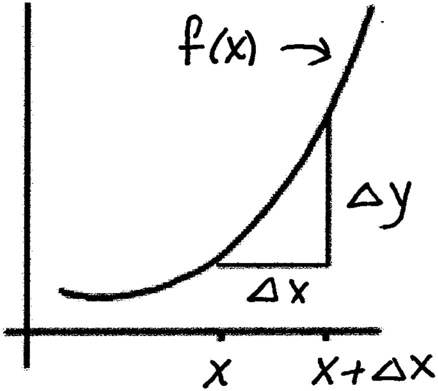
- e.g. if $f(x) = \sqrt{x}$ and a is given as 4

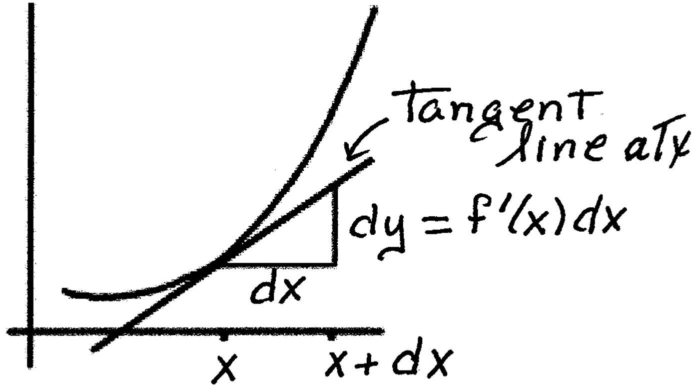
  $f'(x) = \frac{1}{2\sqrt{x}}$, so $f'(4) = \frac{1}{4}$

  $L(x) = 2 + \frac{1}{4}(x-4)$

  & $\sqrt{x} \approx 2 + \frac{1}{4}(x-4)$ for x near 4.

Differential Notation





$\Delta x = dx$ means a change in x

It produces:

$\Delta y = f(x+\Delta x) - f(x)$, the actual change in y or f

$\boxed{dy = f'(x)\,dx}$, the change along the tangent line

Linear approx. tells us $\underset{\text{actual change}}{\Delta y} \approx \underset{\text{approx. change}}{dy}$ when dx is small

dy is called the differential of y

Example – Estimating Maximum Error

The radius of a hemispherical mold is measured at 8 cm with an estimated possible error of .12 cm. Use differentials to estimate
(a) the maximum possible error in the calculated surface area
(b) the relative error.

Relate radius r to surface area A

$A = 2\pi r^2$ (2 because hemisphere)

Error (= change) in r is $dr = \pm .12$

Error in A is ΔA, which we approx. by dA

$$\Delta A \approx dA = A' dr = 4\pi r \, dr$$

When $r = 8$ & $dr = \pm .12$, $dA = (32\pi)(\pm .12) = \pm 12.06 \text{ cm}^2$

(a) max. possible error $\approx \pm 12.06 \text{ cm}^2$

(b) estimated relative error is $\dfrac{dA}{A} = \dfrac{4\pi r \, dr}{2\pi r^2} = 2\dfrac{dr}{r} = \pm .03$

($\pm 3\%$)