The Chain Rule

Section 3.5

The Chain Rule

- enables us to differentiate the composition of functions

$$\frac{d}{dx} f(g(x)) = [f'(g(x))]\, g'(x)$$

- it is a product of derivatives, but note that they are evaluated at different points

- another useful way to write the chain rule:

set $u = g(x)$

$$\boxed{\frac{df}{dx} = \frac{df}{du}\,\frac{du}{dx}}$$

- f is the "outside" function, g the "inside" one

Differentiate outside function first, then inside one.

Motivation for Chain Rule

- if A changes 5 times as quickly as B,

  and if B changes 3 times as quickly as C,

  then A changes 15 times as quickly as C

if $\frac{dA}{dB} = 5$ & $\frac{dB}{dC} = 3$,

then $\frac{dA}{dC} = \frac{dA}{dB}\frac{dB}{dC} = 15$

(rates multiply)

Using the Chain Rule

- $y = (x^2+1)^3$ has the form $\square^3$ where $\square$ is x^2+1

  The cube function is outside.

  The chain rule tells us

  $$\frac{dy}{dx} = [3\,\square^2]\,\square'$$

$$\frac{dy}{dx} = 3(x^2+1)^2(2x)$$

$$= 6x(x^2+1)^2$$

Example – Chain Rule

- Find $f'(t)$ when $f(t) = \dfrac{1}{1-4t^3}$.

(Think $f = \square^{-1}$, where $\square = 1-4t^3$,

so $f' = [-1 \, \square^{-2}] \, \square'$)

$$f'(t) = -(1-4t^3)^{-2}(0-12t^2)$$

$$= \frac{12t^2}{(1-4t^3)^2}$$

Example – Chain Rule

- Find $f'(x)$ when $f(x)=\left(\frac{x-1}{x+1}\right)^2$.

(*think* $f = \square^2$, so $f' = [2\square]\,\square'$)

$$f'(x) = \left[2\left(\frac{x-1}{x+1}\right)\right]\left(\frac{x-1}{x+1}\right)'$$

need quotient rule $\left(\frac{u}{v}\right)' = \frac{u'v-uv'}{v^2}$

$$= 2\left(\frac{x-1}{x+1}\right)\frac{1(x+1)-(x-1)1}{(x+1)^2}$$

$$= 2\left(\frac{x-1}{x+1}\right)\frac{2}{(x+1)^2} = \frac{4(x-1)}{(x+1)^3}$$

Example – Chain Rule

- Find $\frac{d}{dx}a^x$ when a is a positive constant.

(need to know: a^x is defined as $e^{x \ln a}$)

$$\frac{d}{dx} a^x = \frac{d}{dx} e^{x \ln a}$$

(think: $e^{\square}$)

$$= e^{x \ln a} (x \ln a)'$$

(ln a is just a constant)

$$= e^{x \ln a} \ln a$$

$$= a^x \ln a$$

so: $\boxed{\frac{d}{dx} a^x = a^x \ln a}$