Implicit Differentiation

Section 3.6

Implicit Differentiation

- If $x^3 + y^3 = 9xy$, find $\dfrac{dy}{dx}$ at (2, 4).

We can't solve for y and then differentiate.

However, we can assume that y is a function of x near (2, 4). Then for any function of y – e.g. $f(y) = f(y(x))$ – the chain rule gives

$$\frac{df}{dx} = \frac{df}{dy}\frac{dy}{dx} \qquad \text{e.g. } \frac{d}{dx}y^3 = 3y^2\frac{dy}{dx}$$

- Method: Differentiate the given equation with respect to x, and then solve for $\dfrac{dy}{dx}$.

Example – Implicit Differentiation

- If $x^3 + y^3 = 9xy$, find $\frac{dy}{dx}$ at (2, 4).

needs product rule

$$3x^2 + 3y^2\frac{dy}{dx} = 9y + 9x\frac{dy}{dx}$$

$$(3y^2 - 9x)\frac{dy}{dx} = 9y - 3x^2$$ (solve for $\frac{dy}{dx}$)

$$\frac{dy}{dx} = \frac{3y - x^2}{y^2 - 3x}$$

(sub in $x=2, y=4$)

At (2,4) $\frac{dy}{dx} = \frac{12-4}{16-6} = \frac{4}{5}$.

Example – Implicit Differentiation

- Find y' if $y^2 = x^2 + \sin xy$.

$$2yy' = 2x + (\cos xy)(1y + xy')$$

$$= 2x + y\cos xy + xy'\cos xy$$

$$(2y - x\cos xy)\,y' = 2x + y\cos xy$$

$$y' = \frac{2x + y\cos xy}{2y - x\cos xy}$$

(we used $(\sin xy)' = (\cos xy)\underbrace{(xy)'}_{\text{product rule}}$)

Derivative of the Inverse Sine

The graph of $\sin^{-1} x$ is the reflection of a piece of the graph of $\sin x$ across the 45° line $y = x$. So a tangent line at a point of $\sin x$ is reflected to a tangent line at the image point on $\sin^{-1} x$.

- Wanted: y' when $y = \sin^{-1} x$
- Start with: $y = \sin^{-1} x$ means $\underbrace{\sin y = x}$ and $-\pi/2 \le y \le \pi/2$.

$$(\cos y)\, y' = 1$$

$$y' = \frac{1}{\cos y} = \frac{1}{\pm\sqrt{1-\sin^2 y}} \qquad \left(\begin{array}{l} \cos y \text{ is } + \\ \sin y = x \end{array}\right)$$

$$y' = \frac{1}{\sqrt{1-x^2}}$$

Inverse Trig Derivative Formulas

$$\frac{d}{dx}\sin^{-1}x = \frac{1}{\sqrt{1-x^2}}$$

$$\frac{d}{dx}\cos^{-1}x = \frac{-1}{\sqrt{1-x^2}}$$

$$\frac{d}{dx}\mathrm{Tan}^{-1}x = \frac{1}{1+x^2}$$

$$\frac{d}{dx}\cot^{-1}x = \frac{-1}{1+x^2}$$

$$\frac{d}{dx}\sec^{-1}x = \frac{1}{x\sqrt{x^2-1}}$$

$$\frac{d}{dx}\csc^{-1}x = \frac{-1}{x\sqrt{x^2-1}}$$

MEMORIZE!!

Examples – Inverse Trig Differentiation

- Find y' when $y = \tan^{-1}(x^4)$.

$$\left(\underset{c}{y} = \operatorname{Tan}^{-1} \square \,,\ \text{so } y' = \frac{1}{1+\square^2}\, \square' \right)$$

$$y' = \frac{1}{1+(x^4)^2}(4x^3) = \frac{4x^3}{1+x^8}$$

- Find y' when $y = e^{3x}\left(\cos^{-1} x\right)^2$.

$$\underbrace{e^{3x}\left(\cos^{-1} x\right)^2}_{\text{Product Rule}}$$

$$y' = e^{3x}\, 3\, (\cos^{-1} x)^2 + e^{3x}(2\cos^{-1} x)\left(\frac{-1}{\sqrt{1-x^2}} \right)$$

$$= 3e^{3x}(\cos^{-1} x)^2 - \frac{2e^{3x}\cos^{-1} x}{\sqrt{1-x^2}}$$