Indeterminate Forms and L'Hospital's Rule

Section 4.4

L'Hospital's Rule

Assume $f'(x)$ & $g'(x)$ exist, and $g'(x) \neq 0$ near a, except possibly at a. If $\lim_{x \to a} f(x)$ & $\lim_{x \to a} g(x)$ both $= 0$ (or both = either $+\infty$ or $-\infty$)

then
$$\lim_{x \to a} \frac{f(x)}{g(x)} = \lim_{x \to a} \frac{f'(x)}{g'(x)}$$

provided the limit on the right exists.

- $+\infty$ or $-\infty$ are allowable limits on the right.

- a can be finite, $+\infty$ or $-\infty$ or one-sided (a^+ or a^-)

Using L'Hospital's Rule

- Given the problem: find $\lim_{x \to a} \frac{f(x)}{g(x)}$

  We start by looking at $\dfrac{\lim_{x \to a} f(x)}{\lim_{x \to a} g(x)}$ — if this turns out to be one of the indeterminate forms "$\frac{0}{0}$", "$\frac{\infty}{\infty}$" or "$-\frac{\infty}{\infty}$", then we can apply L'H. Rule.

- e.g. $\lim_{x \to 0} \frac{\sin x}{4x} \underset{\text{"}\frac{0}{0}\text{"}}{\overset{L'H}{=}} \lim_{x \to 0} \frac{\cos x}{4} = \frac{1}{4}$

Note: we did <u>not</u> use <u>a quotient rule</u>.

L'H. Examples

- $\lim_{x\to 0} \frac{1-e^x}{\sin x} \underset{\text{"}\frac{0}{0}\text{"}}{\overset{L'H.}{=}} \lim_{x\to 0} \frac{-e^x}{\cos x} = -\frac{1}{1} = -1$

- $\lim_{x\to 0} \frac{x-\sin x}{x^2} \underset{\text{"}\frac{0}{0}\text{"}}{\overset{L'H.}{=}} \lim_{x\to 0} \frac{1-\cos x}{2x} \underset{\text{"}\frac{0}{0}\text{"}}{\overset{L'H.}{=}} \lim_{x\to 0} \frac{\sin x}{2}$

  $= \frac{0}{2} = 0$

- $\lim_{x\to \infty} \frac{\ln x}{x^2} \underset{\text{"}\frac{\infty}{\infty}\text{"}}{\overset{L'H}{=}} \lim_{x\to \infty} \frac{1/x}{2x} = \frac{0}{\infty} = 0$

  or simplify to $\frac{1}{2x^2}$ first.

Indeterminate Forms

We use L'H. Rule directly for limits of the form "$\frac{0}{0}$" or "$\frac{\pm\infty}{\infty}$"

Other indeterminate forms:

$$\underbrace{\text{"}(0)(\pm\infty)\text{"}, \text{"}\infty-\infty\text{"}}, \underbrace{\text{"}0^0\text{"}, \text{"}0^\infty\text{"}, \text{"}\infty^0\text{"}, \text{"}1^\infty\text{"}}$$

- rewrite to "$\frac{0}{0}$" or "$\frac{\pm\infty}{\infty}$" form first.

- occur in $\lim f(x)^{g(x)}$
- we will start these problems by taking ln first.

Another L'H. Example

- $\lim_{x\to\infty} x\ln(1-\frac{1}{x})$ $= \lim_{x\to\infty} \frac{\ln(1-1/x)}{(1/x)}$ Now: "$\frac{0}{0}$"

"$\infty \cdot 0$"

$$\overset{L'H.}{=} \lim_{x\to\infty} \frac{\frac{1}{(1-1/x)}\cancel{\left(+\frac{1}{x^2}\right)}}{\cancel{\left(-\frac{1}{x^2}\right)}}$$

$$= \underbrace{\frac{1}{(1-0)}}_{-1} = -1$$

Another L'H. Example

- $\lim_{x \to \infty} x^{1/x}$

"∞^0" (ln will simplify this)

let $y = x^{1/x}$ (want $\lim\limits_{x\to\infty} y$)

$\ln y = \frac{1}{x} \ln x$ (ln both sides)

$$\lim_{x\to\infty} \ln y = \lim_{x\to\infty} \frac{\ln x}{x} \overset{L'H}{=} \lim_{x\to\infty} \frac{(1/x)}{1} = 0$$

"$\frac{\infty}{\infty}$"

so: As $x \to \infty$, $\ln y \to 0$

& $y = e^{\ln y} \to e^0 = 1$ ⇐ Answer