Optimization Problems

Section 4.7
Tests for Absolute Max/Min

A — In section 4.1 we learned that we can find the absolute max and min values of $f$ on a closed interval $[a,b]$ by listing values of $f$ at all critical numbers and at the endpoints $a$ and $b$.

If we are not working on a closed interval, we cannot use the previous technique — but we may be able to use the following test:

B — Assume $f$ is continuous on an interval $I$, and assume that $f$ has only one critical number (call it $c$).

If:

- $f' > 0$ at $c$:
  - Then $f(c)$ is the absolute max of $f$ on $I$

- $f' < 0$ at $c$:
  - Then $f(c)$ is the absolute min.

In this section, you must use test A or B to verify your answer.
Example 1

- A farmer has 600 ft of fencing with which to construct 3 sides of a rectangular pen. An existing wall forms the 4\textsuperscript{th} side. What dimensions will maximize the enclosed area?

1 - Draw picture — label variables

2 - We want

\[
\text{maximize } A_{\text{area}} \\
2x + y = 600 \\
A = xy
\]
Example 1 – continued

3 – Set up math problem

maximize a function of one variable on an interval

\[
\begin{align*}
\text{maximize } A &= xy \\
&= x(600-2x) \\
\end{align*}
\]

\[
\begin{align*}
\text{on the interval } \\
0 &\leq x \leq 300 \\
\end{align*}
\]

\[
\begin{align*}
\text{since } 2x+y &= 600 \\
y &= 600-2x \\
\end{align*}
\]

\[
\begin{align*}
\text{(since } 2x \leq 600) \\
\end{align*}
\]

4 – Apply calculus to solve the math problem.
Example 1 – concluded

Our math problem now is:

Maximize \( A = x(600 - 2x) \) for \( 0 \leq x \leq 300 \).

\[
A' = 600 - 4x
\]

\[
= 0 \text{ when } x = 150
\]

Test for absolute max

\[
\begin{array}{c}
A \rightarrow 150 \quad A' \rightarrow 300 \\
\text{abs. max}
\end{array}
\]

\[
\begin{array}{c}
0 \quad A' > 0 \quad A' < 0
\end{array}
\]

5 – Answer the ? posed in the original problem.

The dimensions that maximize area are

\[
\begin{align*}
x &= 150 \text{ ft.} \\
y &= 600 - 2x = 300 \text{ ft.}
\end{align*}
\]
Example 2 – Minimizing Distance

- Find the point on the line \( y = 1 - 2x \) that is closest to the point \((2,5)\).

\[
\minimize D = \sqrt{(2-x)^2 + (5-y)^2}
\]

know: \( y = 1 - 2x \)

so \( D = \sqrt{(2-x)^2 + (4+2x)^2} \)

for \(-\infty < x < \infty\)

\( (D' \text{ is going to be messy, because of the } \sqrt{\cdot} \) \)

Notice: \( D \) will be minimized at the same \( x \)-value that minimizes \( D^2 \).

so: \( \text{Set } f = D^2, \ \text{\# minimize } f. \)
Example 2 – concluded

Our math problem is now:

Minimize \( f(x) = (2 - x)^2 + (4 + 2x)^2 \) on \((-\infty, \infty)\)

\[
f'(x) = -2(2-x) + 4(4+2x)
   = -4 + 2x + 16 + 8x = 12 + 10x
\]

\(f' = 0\) when \(x = -1.2\)

Test for absolute min.

Conclusion:

D is minimized when
\(x = -1.2\) \(\Rightarrow y = 1 - 2x = 3.4\)

(If you were asked for value of D, remember that \(D = f''\).)