Newton's Method

Section 4.9

About Newton's Method

- Newton's Method is used to numerically approximate roots of equations of the form $f(x) = 0$.

- We start with an initial estimate x_0 for the root, then use the Newton's Method formula to generate a better approximation x_1; & then use x_1 to generate an even better approximation x_2, etc.

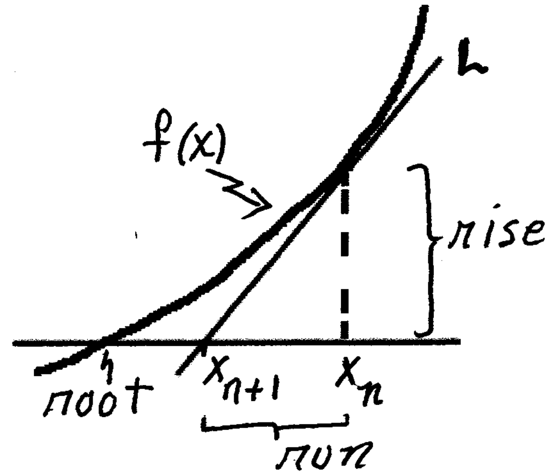
- The method generates a sequence (list) $x_0, x_1, x_2, \ldots$ that, we hope, converges to a root of $f(x) = 0$.

- If we want a root accurate to k decimal places, we can stop when successive approximations x_n and x_{n+1} agree in the first k decimal places.

Newton's Method Formula for x_{n+1}

After we have the n^{th} approximation x_n, we use it to generate the next approximation x_{n+1} —

x_{n+1} will be the point where the tangent line L at x_n crosses the x-axis.



slope of $L = f'(x_n)$

i.e. $= \dfrac{\text{rise}}{\text{run}} = \dfrac{f(x_n)}{x_n - x_{n+1}}$

so: $f'(x_n) = \dfrac{f(x_n)}{x_n - x_{n+1}}$

Solve for x_{n+1}:

$$\boxed{x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}}$$

Example – Newton's Method

- Find a solution between $x=1$ and $x=2$ of the equation $x^4 = x+1$, accurate to 4 decimal places.
  - write as $f(x) = 0$ $\qquad f(x) = x^4 - x - 1 = 0$
  - choose initial estimate x_0 $\qquad x_0 = 1.5$

n	x_n	$f(x_n)$	$f'(x_n)$	x_{n+1}
0	1.5	2.5625	12.5	1.295
1	1.295	.51741	7.6870	1.2277
2	1.2277			1.2208
3	1.2208			1.2207
4	1.2207			1.2207

last column:

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$$

Comments on Newton's Method

- First write as $f(x)=0$.

- Choose initial estimate x_0 carefully, generally from a rough sketch of the graph of f. (Your estimate should have no more than one decimal place.)
  e.g. For all the roots of $5\cos x + x = 0$, a plot shows roots near –2, +2 and 4. You would apply Newton's method for each of these initial estimates. (Be sure to use radian mode.)

- Show your calculations in table form – work across each row of the table, and stop when the entries in the last row agree to the desired # of decimal places.

- You are looking for a solution of $f(x)=0$, so values in the $f(x_n)$ column ~~row~~ should go to zero – if they don't, either you made a bad initial estimate (change it !!) or your calculations are wrong.