Derivatives for Parametric Curves

- If a curve is given by $x = f(t),\ \ y = g(t),$ we calculate the slope with a chain rule:

$$\boxed{\frac{dy}{dx} = \frac{(dy/dt)}{(dx/dt)}}$$

OR $\dfrac{dy}{dx} = \dfrac{g'(t)}{f'(t)}$

- To get $\dfrac{d^2y}{dx^2}$ we use a chain rule again:

$$\boxed{\frac{d^2y}{dx^2} = \frac{\frac{d}{dt}\left(\frac{dy}{dx}\right)}{(dx/dt)}}$$

- Note: $'$ refers to t-derivative.

Example – Finding Derivatives

- Find $\frac{dy}{dx}$ and $\frac{d^2y}{dx^2}$ for the parametric curve $x = t^2,\ y = t^3 - 3t$.

$$\frac{dy}{dx} = \frac{(dy/dt)}{(dx/dt)} = \frac{3t^2-3}{2t} = \frac{3}{2}\left(t - \frac{1}{t}\right)$$

$$\frac{d^2y}{dx^2} = \frac{\frac{d}{dt}\left(\frac{dy}{dx}\right)}{(dx/dt)} = \frac{\frac{3}{2}\left(1+\frac{1}{t^2}\right)}{2t} = \frac{3}{4}\left(\frac{1}{t}+\frac{1}{t^3}\right)$$

Example – Curve Sketching

- For the curve $x = t^2,\ y = t^3 - 3t$, find where the tangent is horizontal or vertical, find the t-intervals where the curve rises or falls, then sketch the curve.

Tangent: slope $= \dfrac{dy}{dx} = \dfrac{(dy/dt)}{(dx/dt)} = \dfrac{3t^2-3}{2t} = \dfrac{3}{2}\left(t - \dfrac{1}{t}\right)$

horiz. Tangent: $\frac{3}{2}\left(t - \frac{1}{t}\right) = 0$, $\left\{\begin{array}{l} t=1 \\ x=1 \\ y=-2 \end{array}\right\}$ or $\left\{\begin{array}{l} t=-1 \\ x=1 \\ y=2 \end{array}\right\}$

Tangent vertical (undefined) when $\left\{\begin{array}{l} t=0 \\ x=0 \\ y=0 \end{array}\right\}$

(we now know 3 important points on the curve)

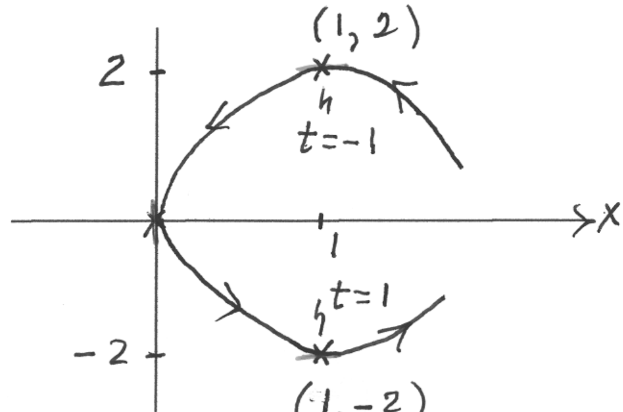
$\Rightarrow$ next slide.

Example – Curve Sketching (concluded)

First derivative analysis :

$$\frac{dx}{dt} = 2t, \quad \frac{dy}{dt} = 3(t^2 - 1)$$

		$(1,2)$ -1		$(0,0)$ 0		$(1,-2)$ 1	
dx/dt	<0		<0		>0		>0
dy/dt	>0		<0		<0		>0
x	$\leftarrow$		$\leftarrow$		$\rightarrow$		$\rightarrow$
y	$\uparrow$		$\downarrow$		$\downarrow$		$\uparrow$
curve	$\nwarrow$		$\swarrow$		$\searrow$		$\nearrow$



Area Under a Parametric Curve

If C is given by $x = f(t),\ \ y = g(t)$ is traversed once as t increases from a to b, then we can easily adapt our previous fomula for the area between C and the x-axis -

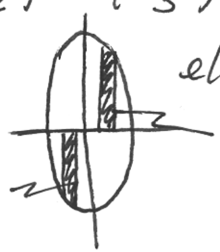
$$\text{Area} = \int_{?_0}^{?_0} |y\,dx| \quad \text{becomes} \quad A = \int_a^b |g(t)\,f'(t)|\,dt$$

(we need $|\cdot|$ because f' may not be positive)

- e. g. Find the area inside the parametric curve
  $x = 2\cos t,\ y = 3\sin t$ for $0 \le t \le 2\pi$.

picture:

$$\left(\frac{x}{2}\right)^2 + \left(\frac{y}{3}\right)^2 = 1$$



$$dA = |y\,dx| = |(3\sin t)(-2\sin t)\,dt|$$
$$= 6\sin^2 t\,dt$$

$$\text{Area} = 6\int_0^{2\pi} \sin^2 t\,dt = 3\int_0^{2\pi}(1-\cos 2t)\,dt$$
$$= 3\left[t - \tfrac{1}{2}\sin 2t\right] = 6\pi$$