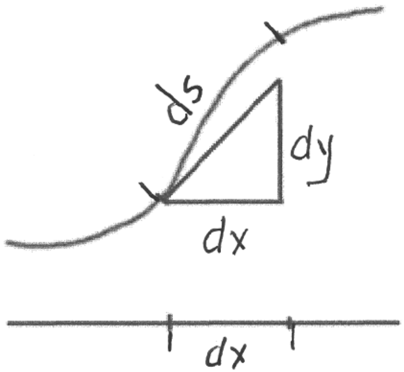
Arc Length for a Parametric Curve

Assume the smooth curve C is traversed exactly once as t increases from a to b.



The differential of arc length, *ds*, is:

$$ds = \sqrt{(dx)^2 + (dy)^2}$$

so $$\boxed{ds = \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt}$$

(now ds will be entirely in Terms of t)

$$\text{Length} = \int_{t=a}^{t=b} ds$$

Example – Arc Length

Find the length of the curve $x = e^{2t}\cos t,\ y = e^{2t}\sin t$ for $0 \le t \le \pi/2$.

$$ds = \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\,dt$$

$$\frac{dx}{dt} = 2e^{2t}\cos t - e^{2t}\sin t \qquad \frac{dy}{dt} = 2e^{2t}\sin t + e^{2t}\cos t$$

$$\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 = e^{4t}\left[4\cos^2 t \cancel{- 4\cos t\sin t} + \sin^2 t\right]$$

$$+\ e^{4t}\left[4\sin^2 t \cancel{+ 4\sin t\cos t} + \cos^2 t\right]$$

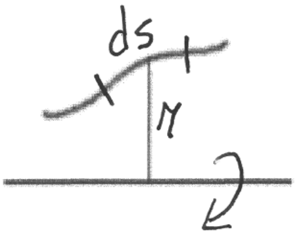
$$= e^{4t}\left[5\cos^2 t + 5\sin^2 t\right] = 5e^{4t}$$

$$ds = \sqrt{5}\,e^{2t}$$

$$\text{Length} = \sqrt{5}\int_0^{\pi/2} e^{2t}\,dt = \frac{\sqrt{5}}{2}\left[e^{\pi} - 1\right]$$

Surface Area (of Revolution)

If the smooth parametric curve C is rotated about an axis, the differential of surface area, dS, is



circumference $= 2\pi r$

$$dS = 2\pi \underbrace{r}_{\text{radius}} \underbrace{ds}_{\text{arc length}}$$

where we must express r and ds in terms of the parameter.

(a diagram helps you to set up r correctly)

Example – Surface Area

- The circle $(x-b)^2 + y^2 = a^2$ is given by the parametric equations

$$x = b + a\cos t,\ \ y = a\sin t, \quad 0 \le t \le 2\pi.$$

Assume $a < b$, and find the surface area obtained by revolving around the y-axis.

$$dS = 2\pi r\, ds$$

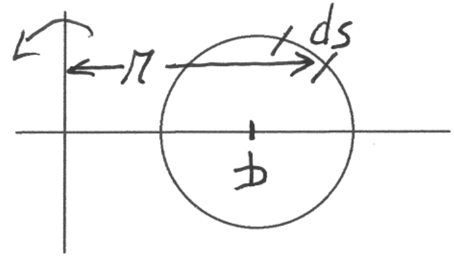
$$r = x = b + a\cos t$$

$$ds = \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt$$

$$= \sqrt{(-a\sin t)^2 + (a\cos t)^2}\, dt$$

$$= \sqrt{a^2(\sin^2 t + \cos^2 t)}\, dt = a\, dt$$

$$S_{\substack{\text{surf.}\\ \text{area}}} = 2\pi a \int_0^{2\pi} (b + a\cos t)\, dt = 4\pi^2 ab$$



$r = x$

revolves to a Torus or donut.