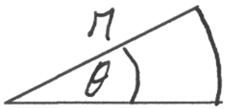
dA in Polar Coordinates

- We start with the area of a sector of a circle:

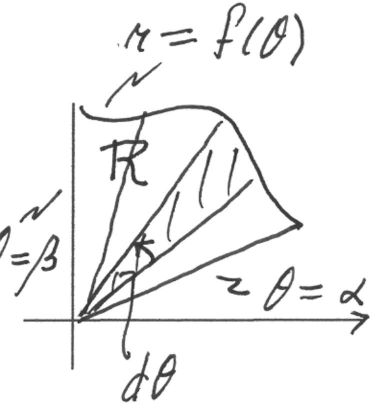


has area $\left(\frac{\theta}{2\pi}\right)$ Times (πr^2) $= \frac{1}{2} r^2 \theta$

fraction of full ○ area of full ○

- Wanted: Area of the region R.

Divide R into subregions by slicing along rays θ = constant.
Let $d\theta$ be the angle between adjacent rays.



Area of slice $\approx \frac{1}{2} r^2 d\theta$

& so we get the differential dArea:

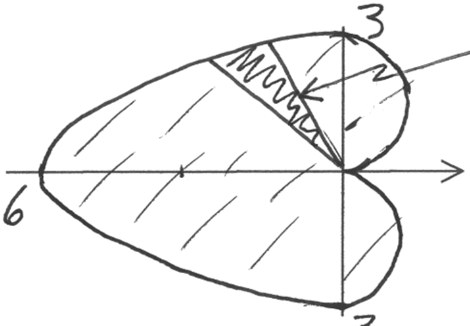
$$\boxed{\begin{array}{l} dA = \frac{1}{2} r^2 d\theta \\ \text{with } r = f(\theta) \end{array}}$$

& $A = \int_{\theta=\alpha}^{\theta=\beta} dA$

Example 1 - Area Inside Cardiod

- Find the area inside the cardiod $r = 3 - 3\cos\theta$.

First – sketch the area. (see e. g. 1 from last lecture)



$$dA = \frac{1}{2} r^2 d\theta = \frac{1}{2}(3 - 3\cos\theta)^2 d\theta$$

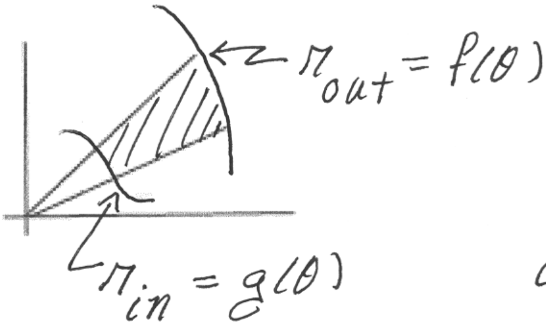
$$= \frac{9}{2}(1 - \cos\theta)^2 d\theta$$

$$A_{rea} = \int_0^{2\pi} \frac{9}{2}(1 - \cos\theta)^2 d\theta$$

note: $0 \le \theta \le 2\pi$ gives 1 copy of cardiod.

or using Top/bottom symmetry: $A = 2\int_0^{\pi} \frac{9}{2}(1 - \cos\theta)^2 d\theta$.

Area Between 2 Polar Curves

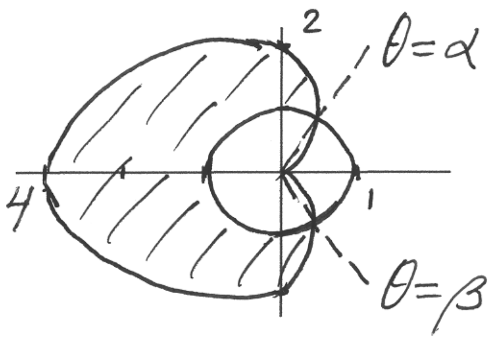


$$dA = \frac{1}{2} r_{out}^2 \, d\theta - \frac{1}{2} r_{in}^2 \, d\theta$$

i.e. $$\boxed{dA = \frac{1}{2}\left[r_{out}^2 - r_{in}^2 \right] d\theta}$$

Example – Area Between Curves

- Find the area outside the circle $r = 1$ and inside the cardiod $r = 2 - 2\cos\theta$.



- Need θ's where curves intersect.

  solve $2-2\cos\theta = 1$; $\cos\theta = 1/2$.

  so $\alpha = \pi/3$ & $\beta = 2\pi - \pi/3$

- We'll use Top/bottom symmetry.

$$A = 2\int_{\pi/3}^{\pi} \frac{1}{2}\left[(2-2\cos\theta)^2 - 1^2\right] d\theta$$

$$= \int_{\pi/3}^{\pi} (3 - 8\cos\theta + 4\cos^2\theta)\, d\theta$$

- Integration limits must always go from lower value to higher.

Length of Polar Curves

- If $r = f(\theta)$, we use the parametric form

$$x = r\cos\theta = f(\theta)\cos\theta \;\; \& \;\; y = f(\theta)\sin\theta.$$

Then $ds = \sqrt{\left(\frac{dx}{d\theta}\right)^2 + \left(\frac{dy}{d\theta}\right)^2}\, d\theta$ SIMPLIFIES TO

$$\boxed{ds = \sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2}\, d\theta} \;\; \&$$

$$\text{Length} = \int_{\theta=\alpha}^{\theta=\beta} ds$$

e. g. Find the length of the cardiod $r = 3 - 3\cos\theta$.

We get one copy of the cardiod for $0 \le \theta \le 2\pi$.

$$L = \int_0^{2\pi} \sqrt{(3-3\cos\theta)^2 + (3\sin\theta)^2}\, d\theta$$

$$= 3\int_0^{2\pi} \sqrt{2 - 2\cos\theta}\, d\theta$$