Sequences

- A sequence is an ordered, unending list of real numbers:

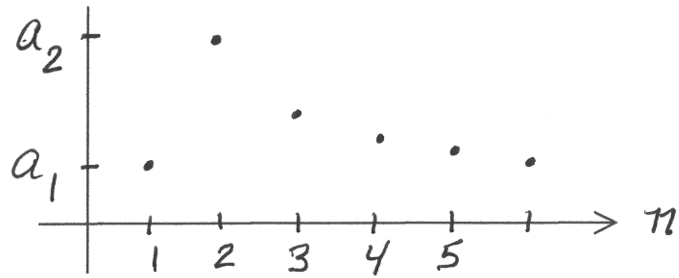
$$a_1, a_2, a_3, \ldots, a_n, \ldots$$

Alternative notation: $\{a_n\}$ or $\{a_n\}_1^\infty$.

a_n is called the n^{th} term.

- It is just a special type of function, with domain restricted to positive integers.

Graph:



- Principal question about a sequence:
  Does it converge as $n \to \infty$, and if so, what is the limit?
  (i.e. does graph have horizontal asymptote?)

Finding Sequence Limits

To find the limit of $\{a_n\}$:

- we can use the limit laws that we learned for functions, including the Squeeze Theorem (section 2.3)
- and techniques for horizontal asymptotes (section 2.6)
- if $a_n = f(n)$, where the function $f(x)$ is defined for large positive x, then l'Hospital's Rule may be useful

A useful fact: If $\lim_{n\to\infty} |a_n| = 0$, then $\lim_{n\to\infty} a_n = 0$.

IMPORTANT – The limit of a convergent sequence must be finite.

Sequence Examples – 1

List the first 4 terms of each sequence $\{a_n\}$.
Does $\{a_n\}$ converge or diverge? If it converges, find the limit.

- $\{a_n\} = \left\{\dfrac{n}{n+1}\right\} = \left\{\dfrac{1}{2}, \dfrac{2}{3}, \dfrac{3}{4}, \dfrac{4}{5}, \ldots\right\}$ (looks like $\lim\limits_{n\to\infty} a_n = 1$)

$$\lim_{n\to\infty} \frac{n}{n+1} = \lim_{n\to\infty} \frac{1}{1+1/n} = \frac{1}{1+0} = 1$$

answer: $\{a_n\}$ converges to 1.

- $\{a_n\} = \left\{\dfrac{e^n}{n^2}\right\} = \left\{e, \dfrac{e^2}{4}, \dfrac{e^3}{9}, \dfrac{e^4}{16}, \ldots\right\}$ (difficult to predict limit)

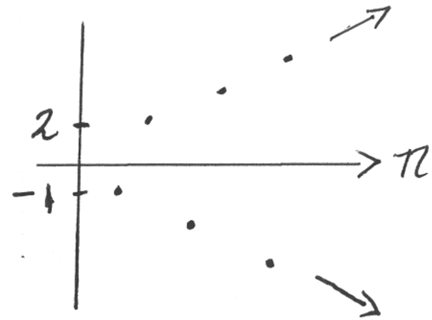
$$\lim_{x\to\infty} \frac{e^x}{x^2} \underset{\text{"}\frac{\infty}{\infty}\text{"}}{\overset{L'H}{=}} \lim_{x\to\infty} \frac{e^x}{2x} \underset{\text{"}\frac{\infty}{\infty}\text{"}}{\overset{L'H}{=}} \lim_{x\to\infty} \frac{e^x}{2} = \infty$$

not a finite limit

answer: $\{a_n\}$ diverges. (To ∞)

Sequence Examples – 2

- $\{a_n\} = \{n\cos n\pi\} = \{\cos\pi,\ 2\cos 2\pi,\ 3\cos 3\pi,\ 4\cos 4\pi, \dots\}$
$$= \{-1, +2, -3, +4, \dots\}$$
pattern shows no finite limit as $n \to \infty$.



answer: $\{a_n\}$ diverges.

- $\{a_n\} = \left\{\dfrac{(-1)^n}{n+2}\right\} = \left\{-\frac{1}{3}, +\frac{1}{4}, -\frac{1}{5}, +\frac{1}{6}, \dots\right\}$ (looks like $\lim\limits_{n\to\infty} a_n = 0$)

$$\lim_{n\to\infty} |a_n| = \lim_{n\to\infty} \frac{1}{n+2} = \frac{1}{\infty} = 0$$

answer: $\{a_n\}$ converges to 0.