Taylor and Maclaurin Series

The Taylor series of f at a (or about a) is

$$\sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!}(x-a)^n$$

$$= f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \frac{f^{(3)}(a)}{3!}(x-a)^3 + \dots$$

- 0! is defined to be 1
- $f^{(n)}(a)$ is the n^{th} derivative of f, evaluated at a.
  $f^{(0)}(a)$ is just $f(a)$.
- When $a = 0$ the series is also called the Maclaurin series.

Taylor Polynomial and Remainder Estimate

- The partial sum $T_n(x) = \sum_{i=0}^{n} \frac{f^{(i)}(a)}{i!}(x-a)^i$ is called a Taylor polynomial. (we'll need to use this in section 11.12)
- $R_n(x) = f(x) - T_n(x)$ is called the remainder. Note that if $\lim_{n\to\infty} R_n(x) = 0$, then the Taylor series converges to $f(x)$.
- We have a remainder estimate (called Taylor's Inequality)

  If $|f^{(n+1)}(x)| \le M$ for $|x-a| < d$, then

  $$|R_n(x)| \le \frac{M}{(n+1)!}|x-a|^{n+1} \quad \text{for } |x-a|<d.$$
- It is not hard to show that $\lim_{n\to\infty} R_n(x) = 0$ when $f(x)$ is e^x or $\sin x$ or $\cos x$. So —

  e^x, $\sin x$, $\cos x$ equal the sum of their Taylor series (for any a)

Example — Maclaurin Series for e^x

- Find the Maclaurin series for e^x (i.e. the Taylor series with $a = 0$).

The series is $\displaystyle\sum_{n=0}^{\infty} \frac{f^{(n)}(0)\,x^n}{n!}$. (we set $a=0$)

$f^{(0)}(x) = e^x \qquad f^{(0)}(0) = e^0 = 1$

$f'(x) = e^x \qquad f'(0) = 1$

$f''(x) = e^x \qquad f''(0) = 1$

$\vdots \qquad\qquad \vdots$

see the pattern: $f^{(n)}(0) = 1$ for every n.

Maclaurin series is $\displaystyle\sum_{n=0}^{\infty} \frac{x^n}{n!}$ — its radius of convergence $R = \infty$,

& by the last comment before this e.g.

$$\boxed{e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}}$$ — REMEMBER THIS SERIES.