Example — Maclaurin Series for $\cos x$

- the series is $\sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} x^n$

$f^{(0)}(x) = \cos x \qquad f^{(0)}(0) = \cos 0 = 1$

same

$f'(x) = -\sin x \qquad f'(0) = 0$

$f''(x) = -\cos x \qquad f''(0) = -1$

$f^{(3)}(x) = \sin x \qquad f^{(3)}(0) = 0$

$f^{(4)}(x) = \cos x \qquad f^{(4)}(0) = 1$

$\vdots \qquad \vdots$

pattern repeats

we see the pattern

$1, 0, -1, 0, 1, 0, -1, \ldots$

Series is $1 + \frac{0x}{1!} - \frac{x^2}{2!} + \frac{0x^3}{3!} + \frac{x^4}{4!} + \frac{0x^5}{5!} - \frac{x^6}{6!} + \cdots$

$= 1 - x^2/2! + x^4/4! - x^6/6! + x^8/8! - \cdots$ converges to $\cos x$

$$\cos x = 1 - x^2/2! + x^4/4! - x^6/6! + \cdots = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{(2n)!}$$

REMEMBER THIS SERIES

Example — a Taylor series

- Find the Taylor series for $\frac{1}{x}$ about $a = 4$.

Series is $\sum_{n=0}^{\infty} \frac{f^{(n)}(4)}{n!}(x-4)^n$

$f^{(0)}(x) = 1/x \qquad f^{(0)}(4) = 1/4$

$f'(x) = -1/x^2 \qquad f'(4) = -1/4^2$

$f''(x) = +2/x^3 \qquad f''(4) = +2/4^3$

$f^{(3)}(x) = -3\cdot 2/x^4 \qquad f^{(3)}(4) = -3\cdot 2/4^4$

$\vdots$

Can you see pattern?

$f^n(4) = \frac{(-1)^n n!}{4^{n+1}}$

Series is: $\frac{1}{4} - \frac{1}{1!4^2}(x-4) + \frac{2}{2!4^3}(x-4)^2 - \frac{3\cdot 2}{3!4^4}(x-4)^3 + \cdots$

$= 1/4 - (x-4)/4^2 + (x-4)^2/4^3 - (x-4)^3/4^4 + \cdots$

OR $\sum_{n=0}^{\infty} (-1)^n (x-4)^n / 4^{n+1}$

[Always calculate at least 4 non-zero terms of series]

Maclaurin Series to Know

$$\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n = 1 + x + x^2 + x^3 + \dots \quad \text{for } |x| < 1 \qquad R = 1$$

$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!} = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \dots \quad \text{for all } x$$

$$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \dots \quad \text{for all } x$$

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \dots \quad \text{for all } x$$

$$R = \infty$$

- Notice that $\cos x$ is an even function, and so its Maclaurin series contains only even powers of x.

  $\sin x$ is an odd function, and so its Maclaurin series contains only odd powers of x.

- We can replace x by polynomial expressions, as well as differentiate or integrate term-by-term to get new Maclaurin series. You can even multiply or divide Maclaurin series.

Examples – More Maclaurin Series

e. g. We can get the Maclaurin series for $\sin x$ from the one for $\cos x$ by differentiation:

$$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \frac{x^8}{8!} - \cdots$$

$$\sin x = -(\cos x)' = -\left(-\frac{2x}{2!} + \frac{4x^3}{4!} - \frac{6x^5}{6!} + \frac{8x^7}{8!} - \cdots\right)$$

$$= x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots$$

e. g. Find the Maclaurin series for e^{-x^2}.

Easiest to do this by replacing x by $(-x^2)$ in series for e^x.

$$e^{-x^2} = 1 + (-x^2) + \frac{(-x^2)^2}{2!} + \frac{(-x^2)^3}{3!} + \cdots$$

$$= 1 - x^2 + \frac{x^4}{2!} - \frac{x^6}{3!} + \cdots$$

(give at least 4 non-0 terms)

Example – Series Approximation of Definite Integral

- Use a series to approximate $\int_0^1 \cos(x^4)\,dx$ with error < 0.001.

$$\cos(x^4) = 1 - \frac{(x^4)^2}{2!} + \frac{(x^4)^4}{4!} - \frac{(x^4)^6}{6!} + \cdots$$

$$= 1 - \frac{x^8}{2!} + \frac{x^{16}}{4!} - \frac{x^{24}}{6!} + \cdots$$

$$\int_0^1 \cos(x^4)\,dx = \left[x - \frac{x^9}{2!\,9} + \frac{x^{17}}{4!\,17} - \frac{x^{25}}{6!\,25} + \cdots\right]_0^1$$

$$= 1 - 1/_{2!\,9} + 1/_{4!\,17} - 1/_{6!\,25} + \cdots$$

$$= 1 - 0.05556 + 0.00245 - 0.00006 + \cdots$$

This is an alternating series – The sizes (i.e. absolute values) of its Terms decrease.

⟹ continued ⟹

Example – Series Approximation of Definite Integral (concluded)

Our remainder estimate for alternating series says that the error after we sum the first n terms is less than the size of the <u>next</u> term.

Since the size of the 4^{th} term is only 0.00006, we conclude that

$$\int_0^1 \cos(x^4)\,dx \approx 1 - 0.05556 + 0.00245 = 0.94689$$

with error < 0.001