The Binomial Series

The Maclaurin series for $(1+x)^k$ (k any real number) is called the binomial series — it is

$$(1+x)^k = 1 + kx + \frac{k(k-1)}{2!}x^2 + \frac{k(k-1)(k-2)}{3!}x^3 + \dots$$

(REMEMBER THIS FORMULA)

- When k is a positive integer, the series only has $(k+1)$ non-zero terms — it is just a polynomial.

  see our first example

- When k is not a positive integer, the radius of convergence $R=1$.

  So we know the series converges for $|x|<1$ & diverges for $|x|>1$. Whether it converges at -1 or $+1$ depends on k.

Example 1

- Expand $(1+x)^4$ using the binomial series formula.

We use $(1+x)^k = 1 + kx + \frac{k(k-1)}{2!}x^2 + \frac{k(k-1)(k-2)}{3!}x^3 + \cdots$

$$(1+x)^4 = 1 + 4x + \frac{4(3)}{2!}x^2 + \frac{4(3)(2)}{3!}x^3 + \frac{4(3)(2)(1)}{4!}x^4$$

$$+ \frac{4(3)(2)(1)(0)}{5!}x^5 + \cdots$$

all these Terms have a 0-factor in the numerator, so They all = 0.

Simplifying –

$$(1+x)^4 = 1 + 4x + 6x^2 + 4x^3 + x^4$$

– just what we'd get by multiplying out –

Example 2

- Use the binomial series to expand $\sqrt{1+x^2}$ as a power series. Find its radius of convergence.

We'll replace x by x^2 in the binomial series, & set $k=1/2$.

$$\sqrt{1+x^2} = 1 + (1/2)(x^2) + \frac{(1/2)(-1/2)}{2!}(x^2)^2 + \frac{(1/2)(-1/2)(-3/2)}{3!}(x^2)^3 + \cdots$$

$$= 1 + \frac{x^2}{2} - \frac{x^4}{8} + \frac{x^6}{16} - \cdots$$

This series will converge for $|x^2|<1$ & diverge for $|x^2|>1$.

i.e. converge for $|x|<1$ & diverge for $|x|>1$

so $R=1$.

Example 3

- Use the binomial series to expand $\dfrac{1}{\sqrt[3]{8+x}}$ as a power series. Find its radius of convergence.

must factor out 8, To Turn this into "1"

$$\frac{1}{\sqrt[3]{8+x}} = \frac{1}{2\sqrt[3]{1+x/8}} = \frac{1}{2}\left(1+x/8\right)^{-1/3}. \qquad \text{So } k=-1/3.$$

$$\frac{1}{\sqrt[3]{8+x}} = \frac{1}{2}\left[1+\left(-\tfrac{1}{3}\right)\left(\tfrac{x}{8}\right)+\frac{\left(-\tfrac{1}{3}\right)\left(-\tfrac{4}{3}\right)}{2!}\left(\tfrac{x}{8}\right)^2 + \frac{\left(-\tfrac{1}{3}\right)\left(-\tfrac{4}{3}\right)\left(-\tfrac{7}{3}\right)}{3!}\left(\tfrac{x}{8}\right)^3+\cdots\right]$$

$$= \frac{1}{2} - \frac{x}{2\cdot 3\cdot 8} + \frac{4x^2}{2\cdot 2!\cdot 3^2\cdot 8^2} - \frac{(4)(7)x^3}{2\cdot 3!\cdot 3^3\cdot 8^3}+\cdots$$

continued

Example 3 - concluded

this series converges for $|x/8| < 1$ & diverges for $|x/8| > 1$

i.e. converges for $|x| < 8$ & diverges for $|x| > 8$.

So $R = 8$.