Taylor Polynomials

The Taylor series of f at a (or about a) is

$$\sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!}(x-a)^n = f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \cdots$$

- The partial sum $T_n(x) = \sum_{i=0}^{n} \frac{f^{(i)}(a)}{i!}(x-a)^i$ is called a Taylor polynomial. ↗ ends with $(x-a)^n$

- Most of the functions we have used in our calculus course are equal to the sum of their Taylor series, within an interval of convergence.
- We can use the simple polynomials $T_n(x)$ to approximate $f(x)$.

  $f(x) \approx T_n(x)$ with remainder $R_n(x) = f(x) - T_n(x)$.

Approximation by Taylor Polynomials

- How good an approximation is $T_n(x)$ to a given $f(x)$?

- We have a remainder estimate (called Taylor's Inequality)

  If $|f^{(n+1)}(x)| \le M$ on an interval $|x-a|<d$,

  then $|R_n(x)| \le \frac{M}{(n+1)!}|x-a|^{n+1}$ on that interval.

- OR if the Taylor series is alternating, we could choose the remainder estimate for an alternating series.

Example 1

- Approximate $\ln x$ by its Taylor polynomial of degree 4 about $a=1$. Use Taylor's Inequality to estimate the accuracy of the approx. on the interval $0.9 \le x \le 1.1$.

$$T_4(x) = f(1) + f'(1)(x-1) + \frac{f''(1)}{2!}(x-1)^2 + \dots + \frac{f^{(4)}(1)}{4!}(x-1)^4$$

$f(x) = \ln x \quad f(1) = 0$

$f'(x) = 1/x \quad f'(1) = 1$

$f''(x) = -1/x^2 \quad f''(1) = -1$

$f^{(3)}(x) = 2/x^3 \quad f^{(3)}(1) = 2$

$f^{(4)}(x) = -6/x^4 \quad f^{(4)}(1) = -6$

So $\ln x \approx (x-1) - \frac{(x-1)^2}{2!} + \frac{2(x-1)^3}{3!} - \frac{6(x-1)^4}{4!}$

or $\ln x \approx (x-1) - (x-1)^2 + \frac{(x-1)^3}{3} - \frac{(x-1)^4}{4}$.

→ continued →

Example 1 - continued

The remainder $R_4(x)$ satisfies $|R_4(x)| \le \frac{M}{5!}|x-1|^5$ when $|f^{(5)}(x)| \le M$.

$f^{(5)}(x) = 24/x^5$. Its max value on $[0.9, 1.1]$ occurs at 0.9.

the max. value of $|x-1|^5$ on $[0.9, 1.1]$ is $(.1)^5$.

So $|R_4(x)| \le \frac{24}{(.9)^5} \frac{(.1)^5}{5!} = 0.000003.$

- If we use the approximation on the larger interval $0.8 \le x \le 1.2$, then

$$|R_4(x)| \le \frac{24}{(.8)^5} \frac{(.2)^5}{5!} = .000195.$$

Example 1 – concluded

Here is a graph of $\ln x$ and $T_4(x)$, with $\ln x$ in red and $T_4(x)$ in blue. It is graphed on the interval $0.05 \le x \le 2.5$.