Infinite Series

- An infinite series is an expression of the form

$$\sum_{n=1}^{\infty} a_n \qquad \text{or} \quad a_1 + a_2 + a_3 + \dots \qquad \text{or} \quad \sum a_n$$

where $\{a_n\}$ is a sequence. We can think of $\sum_{n=1}^{\infty} a_n$ as an attempt to sum the terms of $\{a_n\}$, in order.

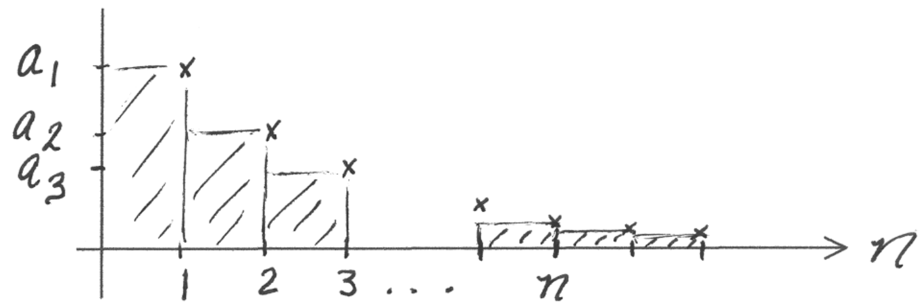
- The n^{th} partial sum of the series is

$$s_n = a_1 + a_2 + \dots + a_n \qquad \left(\text{ends with } a_n\right)$$

- If $\lim_{n\to\infty} s_n = s$ (finite), we say $\sum_{n=1}^{\infty} a_n$ sums (or converges) to s.

Connection with Improper Integrals

$$\sum_{n=1}^{\infty} a_n = a_1 + a_2 + a_3 + \cdots$$



the shaded area should represent

$a_1 + a_2 + a_3 + \ldots$

Shaded area $= \int_0^{\infty}$ (improper) $= \lim_{n\to\infty} \int_0^n = \lim_{n\to\infty} (a_1 + a_2 + \ldots + a_n)$

$= \lim_{n\to\infty} s_n$.

Basic Questions about Series

1 – Does the series $\sum a_n$ converge or diverge?
2 – If $\sum a_n$ converges, can we find its sum?

We can answer both questions easily for a class of series called geometric series, and we'll do this in the next few slides.

Useful facts:

- If $\sum a_n$ and $\sum b_n$ both converge, then

$$\sum (ca_n + db_n) = c\sum a_n + d\sum b_n \qquad \left(\begin{array}{c} c \,\&\, d \\ \text{are} \\ \text{constants} \end{array}\right)$$

- In order for $\sum a_n$ to converge, $\lim_{n\to\infty} a_n$ must be 0.

IMPORTANT

<u>nth Term Test for divergence</u>

If $\lim_{n\to\infty} a_n$ is not 0, then $\sum a_n$ diverges.

Geometric Series – 1

- A <u>geometric series</u> has the form

$$a + ar + ar^2 + ar^3 + \ldots = \sum_{n=0}^{\infty} ar^n$$

a and r are non-zero constants.
r is called the common ratio, and a is called the first term.

- Case 1 - when $|r| \geq 1$

the n^{th} Term $a_n = ar^{n-1}$, so $|a_n| \geq |a|$,

& $\lim_{n \to \infty} a_n$ does not $= 0$.

So, by the n^{th} Term Test, the series $\sum a_n$ diverges.

(e.g. $1 - 2 + 2^2 - 2^3 + \ldots$ has n^{th} Term $a_n = (-2)^{n-1}$

$\lim_{n \to \infty} (-2)^{n-1}$ is not zero – so series diverges)

Geometric Series – 2

- Case 2 - when $|r| < 1$
  We can find a formula for s_n

$$\begin{aligned} s_n &= a + ar + ar^2 + \dots + ar^{n-1} \\ - \quad rs_n &= \phantom{a + {}} ar + ar^2 + \dots + ar^{n-1} + ar^n \\ \hline (1-r)s_n &= a - ar^n \end{aligned}$$

So $s_n = \dfrac{a(1-r^n)}{(1-r)}$

Since $\lim_{n\to\infty} r^n = 0$ when $|r|<1$, we see

$$\lim_{n\to\infty} s_n = \frac{a}{1-r}.$$

- Conclusion: the geometric series $a + ar + ar^2 + \dots$ converges to the sum $\dfrac{a}{1-r}$ when $|r|<1$, and diverges when $|r| \geq 1$.

Examples – Geometric Series

- $\sum_{n=1}^{\infty} \frac{1}{2^n} = \frac{1}{2} + \frac{1}{2^2} + \frac{1}{2^3} + \ldots$ (write out first 3 or 4 Terms)

  a geometric series: $a = 1/2$, $r = 1/2$ $(|r|<1)$

  conclusion: $\sum_{n=1}^{\infty} \frac{1}{2^n}$ converges to the sum $\frac{a}{1-r} = \frac{(1/2)}{1-1/2} = 1.$

- $\sum_{n=1}^{\infty} \left(\frac{2}{3}\right)^n - \frac{5}{4^n} = \frac{2}{3} - \frac{5}{4} + \left(\frac{2}{3}\right)^2 - \frac{5}{4^2} + \left(\frac{2}{3}\right)^3 - \frac{5}{4^3} + \ldots$

  $= \underbrace{\left[\frac{2}{3} + \left(\frac{2}{3}\right)^2 + \left(\frac{2}{3}\right)^3 + \ldots\right]}_{\text{geometric series } a = 2/3,\ r = 2/3} - \underbrace{\left[\frac{5}{4} + \frac{5}{4^2} + \frac{5}{4^3} + \ldots\right]}_{\text{geometric series } a = 5/4,\ r = \frac{1}{4}}$

  Both have $|r|<1$, so both converge.

  conclusion: series converges to the sum $\frac{2/3}{1-2/3} - \frac{5/4}{1-1/4} = 2 - 5/3 = 1/3$

Example – Repeating Decimals

We can use a geometric series expansion to convert a repeating decimal to a rational number.

e.g. $1.121212\ldots$ can be written as

$$\underbrace{1+\frac{12}{100}+\frac{12}{(100)^2}+\frac{12}{(100)^3}+\cdots}_{\text{geometric series}} = 1+\frac{(12/100)}{1-(1/100)}$$

$a = 12/100 \quad r = 1/100$

$$= 1 + 12/99$$

$$= 111/99.$$