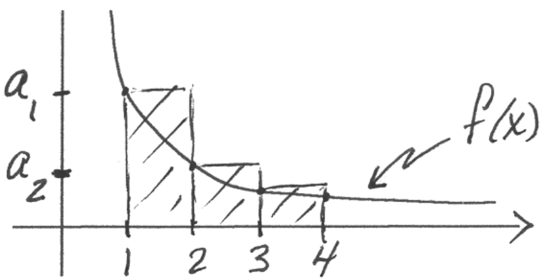
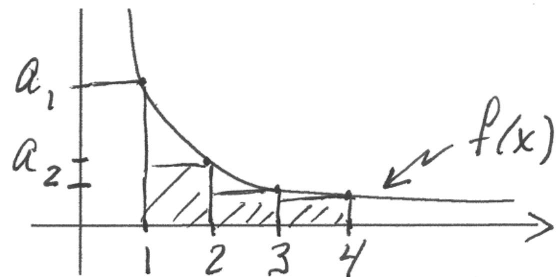
The Integral Test

- Assume $a_n = f(n)$ where f is a continuous, positive, decreasing function on $[1, \infty)$.

  Then $\sum_{n=1}^{\infty} a_n$ and $\int_1^{\infty} f(x)\,dx$ either both converge to finite values, or both diverge.





$$\int_1^{\infty} f(x)\,dx \le a_1 + a_2 + a_3 + \dots \qquad a_1 + a_2 + a_3 + \dots \le a_1 + \int_1^{\infty} f(x)\,dx$$

if right side finite, so is left.

Example 1 – Integral Test

- $\sum_{n=1}^{\infty} \frac{1}{n^2} = 1 + \frac{1}{4} + \frac{1}{9} + \cdots \qquad a_n = f(n), \text{ where } f(x) = \frac{1}{x^2}$

Apply integral Test – first verify hypotheses of $\int$ Test.

$\frac{1}{x^2}$ is positive & continuous for $x \geq 1$ ✓

" " decreasing – see graph

$$\int_1^{\infty} \frac{dx}{x^2} = \lim_{t\to\infty} \left[-\frac{1}{x}\right]_1^t = \lim_{t\to\infty} \left[-\frac{1}{t} + 1\right] = 1$$

The integral converges, so the series $\sum_{n=1}^{\infty} \frac{1}{n^2}$ converges.

- Note: The value of the $\int$ does not = the sum of the series.

Example 2 – Integral Test

$$\sum_{n=1}^{\infty} \frac{n}{e^{n^2}} = \frac{1}{e} + \frac{2}{e^4} + \frac{3}{e^9} + \cdots \qquad f(x) = xe^{-x^2}. \text{ Apply } \int \text{ Test.}$$

f is positive & continuous for $x \geq 1$ ✓

Is f decreasing? It will be if $f' < 0$

$$f' = e^{-x^2} - 2x^2 e^{-x^2} = \underbrace{(1-2x^2)}_{(<0)} \underbrace{e^{-x^2}}_{(>0)} < 0 \text{ for } x \geq 1 ✓$$

$$\int_1^{\infty} x e^{-x^2}\,dx = \lim_{t\to\infty} \left[\frac{e^{-x^2}}{-2} \right]_1^t = \lim_{t\to\infty} \left[\frac{e^{-t^2}}{-2} + \frac{1}{2e} \right] = 1/2e$$

the integral converges, so the series $\sum_{n=1}^{\infty} n/e^{n^2}$ converges.

Example 3 – p-series

$$\sum_{n=1}^{\infty} \frac{1}{n^p} = 1 + \frac{1}{2^p} + \frac{1}{3^p} + \dots \quad (p \text{ a constant})$$

For $p \le 0$, $a_n = \frac{1}{n^p} \ge 1$, so $\lim_{n\to\infty} a_n$ is not zero, &

series diverges by n^{th} Term Test.

For $p > 0$, we apply integral Test with $f(x) = 1/x^p$.

Easy to calculate integral, & we find

> The p-series $\sum 1/n^p$ converges for $p > 1$
> & diverges for $p \le 1$

IMPORTANT

- When $p = 1$, $\sum_{n=1}^{\infty} 1/n$ is called the harmonic series.

It diverges.

Estimating the Sum of a Series

How close is the partial sum s_n to the sum s of a series?

$s - s_n = a_{n+1} + a_{n+2} + \dots$ is called the remainder R_n

- If the Integral Test applies, and Σa_n converges, then

$$\int_{n+1}^{\infty} f(x)\,dx \le R_n \le \int_{n}^{\infty} f(x)\,dx$$

- e. g. How many terms ~~must~~ should we use to estimate the sum of $\sum_{n=1}^{\infty} \frac{1}{n^2}$ to within 0.01?

We want the smallest integer n with $R_n \le 0.01$

$$R_n \le \int_n^{\infty} \frac{dx}{x^2} = \lim_{t \to \infty} \left[-\frac{1}{t} + \frac{1}{n}\right] = \frac{1}{n} \text{ should be } \le 0.01$$

So $n \geq \frac{1}{0.01} = 100$. Answer: we need 100 terms.