Comparison Tests

- Assume that Σa_n and Σb_n have positive terms, and that we know whether or not Σb_n converges.

1. If Σb_n converges & $a_n \le b_n$ for all n, then Σa_n converges.

   (If the bigger series converges, so does the smaller one.)

2. If Σb_n diverges & $a_n \ge b_n$ for all n, then Σa_n diverges.

   (If the smaller series diverges, so does the bigger one.)

3. If $\lim\limits_{n\to\infty} \dfrac{a_n}{b_n} = c$, and $0 < c < \infty$, then Σa_n and Σb_n either both converge or both diverge. (This is the Limit Comparison Test.)

   (my favorite is 3.)

Series to Compare With

To use a comparison test, you compare with the known behavior of a series $\sum b_n$.

Here are examples that we know so far:

1. When $\sum b_n$ is a geometric series, $b_n = ar^n$,

   we know: $\sum b_n$ converges for $|r| < 1$ & diverges for $|r| \geq 1$.

2. When $\sum b_n$ is a p-series, $b_n = 1/n^p$,

   we know: $\sum b_n$ converges for $p > 1$ & diverges for $p \leq 1$.

($p = 1$ gives the harmonic series — it diverges)

Examples – Limit Comparison – 1

- Determine whether $\sum_{n=1}^{\infty} \frac{1}{3^n - 1}$ converges or diverges.

For large n, $a_n = 1/3^n-1$, behaves like $1/3^n$.

We'll use Limit Comparison Test with $\sum b_n = \sum 1/3^n$.

$$\frac{a_n}{b_n} = \frac{(1/3^n-1)}{(1/3^n)} = \frac{3^n}{3^n-1} = \frac{1}{1-\frac{1}{3^n}} \xrightarrow[n\to\infty]{\text{as}} \frac{1}{1-0} = 1.$$

Since $\sum b_n$ is a convergent geometric series with $r = 1/3$, $\sum 1/3^n-1$ also converges.

- Always state what test you use, and state your conclusion.

Examples – Limit Comparison – 2

- Determine whether $\sum_{n=2}^{\infty} \frac{n^2+2}{n^3-1}$ converges or diverges.

For large n, $a_n = \frac{n^2+2}{n^3-1}$ behaves like $\frac{n^2}{n^3} = \frac{1}{n}$.

We'll use Limit Comparison Test with $\sum b_n = \sum 1/n$.

$$\frac{a_n}{b_n} = \left(\frac{n^2+2}{n^3-1}\right)\left(\frac{n}{1}\right) = \frac{n^3+2n}{n^3-1} = \frac{1+2/n^2}{1-1/n^3} \xrightarrow[n\to\infty]{as} 1$$

Since $\sum b_n$ is the divergent harmonic series,

$\sum \frac{n^2+2}{n^3-1}$ also diverges.

- $\sum_{n=1}^{\infty} \frac{1}{\sqrt{n^3+2n}}$ — Try limit comparison with $\sum \frac{1}{n^{3/2}}$.

Examples – Ordinary Comparison

- Determine whether $\sum_{n=1}^{\infty} \frac{1}{n2^n}$ converges or diverges.

$$\sum_{n=1}^{\infty} \frac{1}{n2^n} = \frac{1}{2} + \frac{1}{2\cdot 2^2} + \frac{1}{3\cdot 2^2} + \dots \le \frac{1}{2} + \frac{1}{2^2} + \frac{1}{2^3} + \dots$$

geometric series, $r = 1/2$

$\sum \frac{1}{n2^n}$ converges because it is smaller than a convergent geometric series. (Comparison Test)

- Determine whether $\sum_{n=1}^{\infty} \frac{2^n}{n}$ converges or diverges.

$$\sum_{n=1}^{\infty} \frac{2^n}{n} = 2 + \frac{2^2}{2} + \frac{2^3}{3} + \dots \ge 1 + \frac{1}{2} + \frac{1}{3} + \dots$$ (divergent harmonic series)

$\sum \frac{2^n}{n}$ diverges because it is greater than the divergent harmonic series. (Comparison Test)