Alternating Series

- are series whose terms are alternately positive & negative.

e.g. $1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \cdots$

standard notation $\sum (-1)^n b_n$ or $\sum (-1)^{n+1} b_n$ or $\sum (-1)^{n-1} b_n$

where $b_n > 0$ for all n.

- The Alternating Series Test:

If an alternating series satisfies

(1) $\lim\limits_{n \to \infty} b_n = 0$

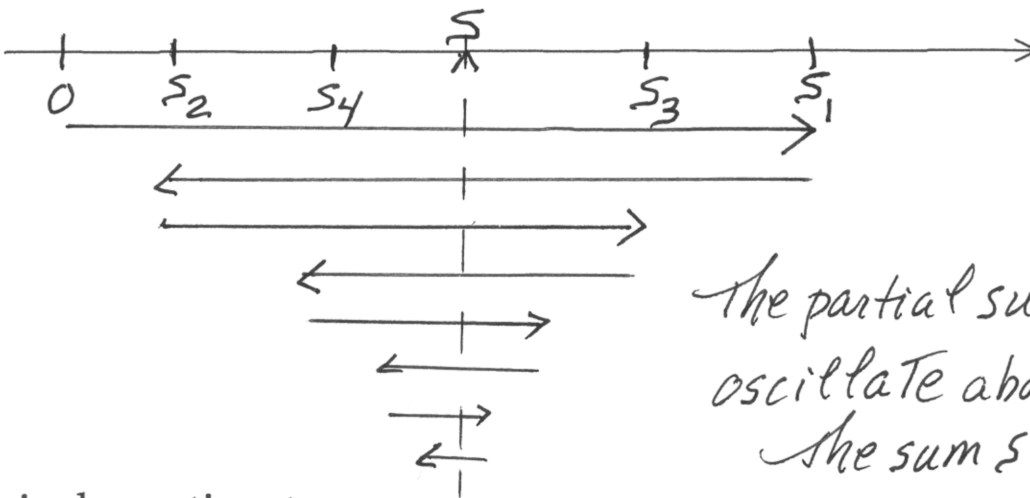
AND (2) $b_{n+1} \leq b_n$ for all n (i.e. b_n's decrease)

then the series converges.

Convergence Picture and Remainder Estimate

$$\sum_{n=1}^{\infty}(-1)^{n+1}b_n = b_1 - b_2 + b_3 - \ldots$$

The n^{th} partial sum is $s_n = b_1 - b_2 + \cdots (\pm) b_n$



the partial sums oscillate about the sum s

- Remainder estimate –

the remainder $R_n = s - s_n$ satisfies $|R_n| \leq b_{n+1}$

Examples – Alternating Series

- Determine whether $\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n}$ converges or diverges.

Series is $1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \cdots$ alternating series; $b_n = \frac{1}{n}$

Check hypotheses of Alt. Series Test: (1) $\lim_{n\to\infty} b_n = \lim_{n\to\infty} \frac{1}{n} = 0$ ✓

(2) $\frac{1}{n+1} \le \frac{1}{n}$ (obvious) for all n ✓

Conclusion: $\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n}$ converges by the Alternating Series Test.

- $\sum_{n=2}^{\infty} \frac{(-1)^n}{\ln n} = \frac{1}{\ln 2} - \frac{1}{\ln 3} + \frac{1}{\ln 4} - \cdots$ alternating series; $b_n = \frac{1}{\ln n}$

(1) $\lim_{n\to\infty} \frac{1}{\ln n} = 0$ ✓ (2) $\frac{1}{\ln(n+1)} \le \frac{1}{\ln n}$ since $\ln(n+1) \ge \ln n$. ✓

Conclusion: The series converges by the Alt. Series Test.

Example – Alternating Series

Note - the condition $b_{n+1} \le b_n$ for all n can be replaced by $b_{n+1} \le b_n$ for all $n \ge N$ where N is some fixed number.

- $\sum_{n=1}^{\infty} (-1)^{n+1} \dfrac{n^2}{n^3+1} = \dfrac{1}{2} - \dfrac{4}{9} + \dfrac{9}{28} - \dfrac{16}{65} + \ldots$ alternating series $b_n = \dfrac{n^2}{n^3+1}$

(1) $\lim_{n\to\infty} \dfrac{n^2}{n^3+1} = \lim_{n\to\infty} \dfrac{1/n}{1+1/n^3} = \dfrac{0}{1+0} = 0$ ✓

(2) Do the b_n's decrease? They will if $f(x) = \dfrac{x^2}{x^3+1}$ decreases.

So – we'll show $f' < 0$.

$$f'(x) = \frac{2x(x^3+1) - x^2(3x^2)}{(x^3+1)^2} = \frac{2x - x^4}{(x^3+1)^2} = \frac{x\overset{*}{(2-x^3)}}{(x^3+1)^2}.$$

$$= \frac{(+)(-)}{(+)} < 0$$

* $2 - x^3 < 0$ when $x > 2^{1/3}$

Conclusion: The series converges by the Alt. Series Test.

Example – Remainder (Error) Estimate

- How many terms should we add in order to find the sum of $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n^3}$ with error < 0.01?

Since $|R_n| \le b_{n+1} = \frac{1}{(n+1)^3}$, we want the smallest integer n satisfying. $\frac{1}{(n+1)^3} < 0.01$

$$(n+1)^3 > \frac{1}{0.01} = 100$$

$$n > (100)^{1/3} - 1 \approx 3.64$$

Answer: $n = 4$ Terms.

We have shown that

$$\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n^3} \approx 1 - \frac{1}{2^3} + \frac{1}{3^3} - \frac{1}{4^3} = .896412$$

with error < 0.01