Absolute and Conditional Convergence - 1

Σa_n is called absolutely convergent when $\Sigma |a_n|$ converges.

- a series with all positive terms always converges absolutely

- some alternating series converge absolutely

  e. g. $\sum\limits_{n=1}^{\infty} \frac{(-1)^{n+1}}{n^2}$ converges absolutely because $\sum\limits_{n=1}^{\infty} \frac{1}{n^2}$ converges.

  e. g. $\sum\limits_{n=1}^{\infty} \frac{(-1)^{n-1}}{n}$ does not converge absolutely, although it does "pass" the Alternating Series Test.

Absolute and Conditional Convergence – 2

Fact: If a series is absolutely convergent, then it is convergent.

i.e. if $\sum |a_n|$ converges, we know that $\sum a_n$ also converges

When $\sum a_n$ converges and $\sum |a_n|$ diverges, we say $\sum a_n$ is conditionally convergent.

e.g. $\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n}$ is conditionally convergent

because $\sum_{n=1}^{\infty} \frac{1}{n}$ diverges (it is the harmonic series)

& $\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n}$ converges (it passes the Alt. Series Test)

The Ratio Test & the Root Test

- Ratio Test: If $\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = L$, then

  (1) when $L < 1$, $\sum a_n$ converges absolutely

  (2) when $1 < L \le \infty$, $\sum a_n$ diverges.

- Root Test: If $\lim_{n\to\infty} |a_n|^{1/n} = L$, then

  (1) when $L < 1$, $\sum a_n$ converges absolutely

  (2) when $1 < L \le \infty$, $\sum a_n$ diverges

- When $L = 1$, both tests are inconclusive — some series with $L=1$ converge & some diverge.

Examples – Ratio and Root Tests – 1

- $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^{n+1} n}{3^n} = \frac{1}{3} - \frac{2}{3^2} + \frac{3}{3^3} - \cdots \qquad |a_n| = \frac{n}{3^n}$

We will apply the Ratio Test:

$$\left| \frac{a_{n+1}}{a_n} \right| = \left(\frac{n+1}{3^{n+1}} \right) \left(\frac{3^n}{n} \right) = \left(\frac{n+1}{n} \right) \left(\frac{3^n}{3^{n+1}} \right)$$

$$= \left(1 + \frac{1}{n} \right) \left(\frac{1}{3} \right) \xrightarrow[n \to \infty]{\text{as}} \frac{1}{3} = L$$

Conclusion: since $L < 1$, our series converges absolutely.

Examples – Ratio and Root Tests – 2

- $\sum_{n=1}^{\infty} \frac{(-1)^n n!}{(2n)!} = -\frac{1}{2!} + \frac{2!}{4!} - \frac{3!}{6!} + \dots = -\frac{1}{2} + \frac{1}{4 \cdot 3} - \frac{1}{6 \cdot 5 \cdot 4} + \dots$

We will apply the Ratio Test. $|a_n| = n!/(2n)!$

$$\left|\frac{a_{n+1}}{a_n}\right| = \frac{(n+1)!}{(2(n+1))!} \frac{(2n)!}{n!} = \frac{(n+1)!}{n!} \frac{(2n)!}{(2n+2)!}$$

$$= \frac{(n+1)\cancel{n!}}{\cancel{n!}} \frac{\cancel{(2n)!}}{(2n+2)(2n+1)\cancel{(2n)!}} = \frac{n+1}{4n^2+6n+2}$$

$$= \frac{1/n + 1/n^2}{4+6/n+2/n^2} \xrightarrow[n\to\infty]{as} \frac{0}{4} = 0 = L$$

Conclusion: since $L<1$, our series converges absolutely.

- Note: Ratio Test is very handy when a_n involves factorials.

Examples – Ratio and Root Tests – 3

Note: Two useful limits to know:

$$\lim_{n\to\infty} a^{1/n} = 1 \text{ for any finite constant } a>0.$$

$$\lim_{n\to\infty} n^{1/n} = 1 \quad \left(\begin{array}{l}\text{proved with} \\ \text{L'Hospital's Rule}\end{array}\right)$$

- $\displaystyle\sum_{n=1}^{\infty} \left(\frac{2}{n}\right)^n = 2 + 1 + \left(\frac{2}{3}\right)^3 + \left(\frac{2}{4}\right)^4 + \cdots$

We'll apply the Root Test: $|a_n| = \left(\frac{2}{n}\right)^n$

$$\lim_{n\to\infty} |a_n|^{1/n} = \lim_{n\to\infty} \frac{2}{n} = 0 = L$$

Conclusion: since $L<1$, our series converges absolutely.

Examples – Ratio and Root Tests – 4

- $\sum_{n=1}^{\infty} \frac{(-1)^{n+1} n}{3^n} = \frac{1}{3} - \frac{2}{3^2} + \frac{3}{3^3} - \cdots \qquad |a_n| = n/3^n$

  (We just did this with the Ratio Test – now we'll do it with the Root Test)

  $$\lim_{n\to\infty} |a_n|^{1/n} = \lim_{n\to\infty} \frac{n^{1/n}}{3} = \frac{1}{3} = L$$

  Conclusion: since $L < 1$, our series converges absolutely.

- $\sum_{n=1}^{\infty} \frac{(-1)^{n+1} 3^n}{n}$ We easily calculate $L = 3$ – giving us the

  Conclusion: since $L > 1$, our series diverges.