Power Series

A power series has the form

$$\sum_{n=0}^{\infty} c_n (x-a)^n = c_0 + c_1(x-a) + c_2(x-a)^2 + \cdots$$

x is a variable
a is a constant (often $a = 0$)
The c_n's are called the coefficients.

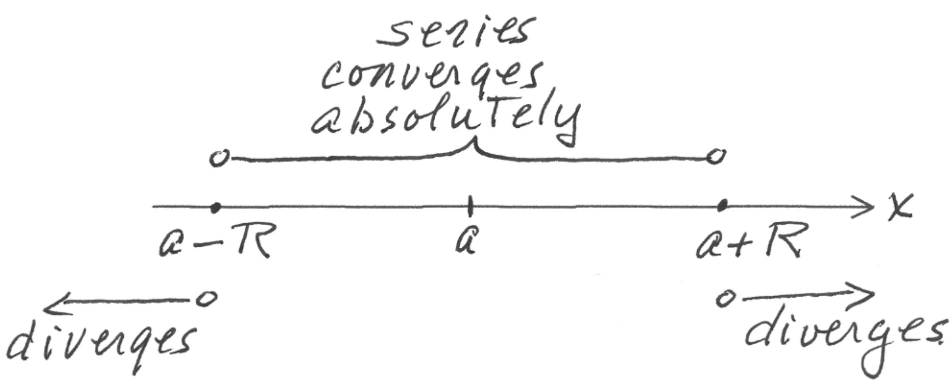
(A power series looks like an "unending polynomial" in powers of $(x-a)$. True polynomials always end.)

- For each power series, there is a number R $(0 \le R \le \infty)$ so that $\sum_{n=0}^{\infty} c_n (x-a)^n$ converges absolutely for $|x-a| < R$ & diverges for $|x-a| > R$.
- R is the radius of convergence. We find R using the ratio or root test.

Interval of Convergence

- The principal question about a power series is:
  For which values of x does the series converge?

- The interval of convergence consists of all x-values for which the series converges.
  It includes $a-R < x < a+R$
  & may include $a-R$ or $a+R$ or both, depending on the specific power series.



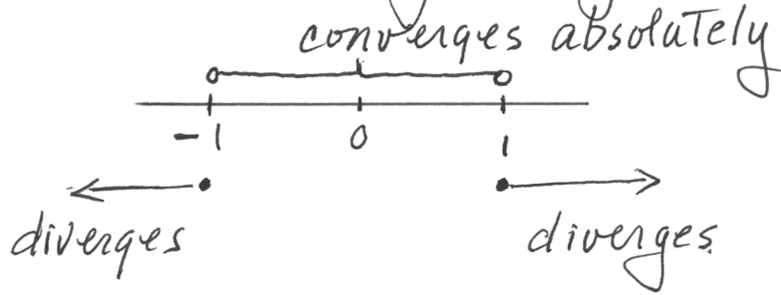
Examples – Power Series -1

- Find R and the interval of convergence of $\sum_{n=0}^{\infty} x^n = 1 + x + x^2 + \ldots$

We already know all about this series — since it is a geometric series with $r = x$.

So: the series converges for $|x| < 1$ & diverges for $|x| \geq 1$.

i.e. $\sum_{n=0}^{\infty} x^n$ has interval of convergence $-1 < x < 1$.



note that $R = 1$

Examples – Power Series –2

- Find R and the interval of convergence of $\sum_{n=1}^{\infty} \frac{(x-2)^n}{n}$.

Series starts as: $(x-2) + \frac{(x-2)^2}{2} + \frac{(x-2)^3}{3} + \dots$

We'll apply ratio Test — $|a_n| = \frac{|x-2|^n}{n}$ (absolute values are important)

$$\left|\frac{a_{n+1}}{a_n}\right| = \frac{|x-2|^{n+1}}{n+1} \frac{n}{|x-2|^n} = |x-2|\left(\frac{n}{n+1}\right) = |x-2|\frac{1}{1+1/n}$$

$$\xrightarrow[n\to\infty]{\text{as}} |x-2| = L.$$

So: our series converges absolutely when $|x-2| < 1$

i.e. $2-1 < x < 2+1$ or $1 < x < 3$.

& it diverges when $|x-2| > 1$ — i.e. $x < 1$ or $x > 3$.

$\underline{R = 1}$

continued →

Examples – Power Series –2 concluded

What happens when $|x-2|=1$ — i.e. when $x=1$ or $x=3$?

These x-values have to be tested separately.

<u>When $x=1$</u> (sub $x=1$ into series)

series is $\sum_{n=1}^{\infty} \frac{(1-2)^n}{n} = \sum_{n=1}^{\infty} \frac{(-1)^n}{n}$. Alternating series —

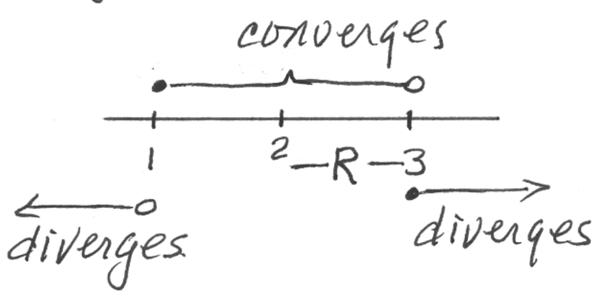
converges since $b_n = \frac{1}{n}$ decreases with $\lim_{n\to\infty} b_n = 0$.

<u>When $x=3$</u>

series is $\sum_{n=1}^{\infty} \frac{(3-2)^n}{n} = \sum_{n=1}^{\infty} \frac{1}{n}$ divergent harmonic series.

Conclusion: $R=1$

Interval of Convergence $1 \leq x < 3$



Examples – Power Series –3

- Find R and the interval of convergence of $\sum_{n=0}^{\infty} \frac{(-1)^n (x-1)^n}{3^n}$.

Series starts: $1 - \frac{(x-1)}{3} + \frac{(x-1)^2}{9} - \dots$

Apply root test: $|a_n| = \frac{|x-1|^n}{3^n}$. $\lim_{n\to\infty} |a_n|^{1/n} = \frac{|x-1|}{3} = L$.

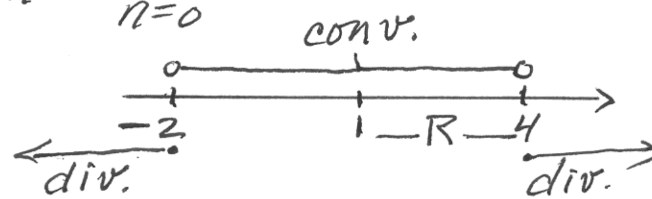
Our series converges absolutely when $\frac{|x-1|}{3} < 1$. $-2 < x < 4$

" series diverges when $x < -2$ or $x > 4$. $\underline{R = 3}$

When $x = -2$, series is $\sum_{n=0}^{\infty} \frac{(-1)^n(-3)^n}{3^n} = \sum_{n=0}^{\infty} 1$. Diverges ($n^{th}$ Term Test)

When $x = 4$, series is $\sum_{n=0}^{\infty} \frac{(-1)^n 3^n}{3^n} = \sum_{n=0}^{\infty} (-1)^n$. ...ditto...

Interval of convergence: $-2 < x < 4$



Examples – Power Series – 4

- Find R and the interval of convergence of $\sum_{n=0}^{\infty} \frac{(x+1)^n}{n!}$.

Series starts: $1 + (x+1) + \frac{(x+1)^2}{2!} + \ldots$

(Note: $0!$ is defined to be 1.)

Apply ratio Test (since we have factorials)

$$\left|\frac{a_{n+1}}{a_n}\right| = \frac{|x+1|^{n+1}}{(n+1)!}\,\frac{n!}{|x+1|^n} = |x+1|\,\frac{\cancel{n!}}{(n+1)\cancel{n!}} = \frac{|x+1|}{n+1}$$

$$\xrightarrow[n\to\infty]{as} 0 = L$$

Conclusion: Since $L=0$, our series converges absolutely for all values of x.

$R = \infty$; Interval of convergence: $-\infty < x < \infty$.