Power Series Representations

The geometric power series

$$\sum_{n=0}^{\infty} x^n = 1 + x + x^2 + \dots = \frac{1}{1-x} \text{ whenever } |x| < 1$$

equals, since the series sums to $1/_{1-x}$.

- We say that $1 + x + x^2 + x^3 + \dots$ is a <u>power series representation</u> for $\frac{1}{1-x}$, with interval of convergence $-1 < x < 1$.

- We can replace the variable x with other expressions to get power series representations for other functions.

Example 1

- Find a power series representation for $\frac{1}{1+x^2}$.

We start with $\frac{1}{1-x} = 1 + x + x^2 + x^3 + \ldots$ for $|x| < 1$.

Replace x by the expression $(-x^2)$: we get

$$\frac{1}{1+x^2} = 1 - x^2 + x^4 - x^6 + \ldots$$

$$\overset{\text{OR}}{=} \sum_{n=0}^{\infty} (-1)^n x^{2n}$$

This representation is valid for $|-x^2| < 1$

i.e. for $|x| < 1$.

(so $R = 1$)

Differentiation or Integration of Power Series

- If a power series $\sum c_n(x-a)^n$ has radius of convergence $R>0$,
  then we can differentiate it term-by-term
  or integrate it term-by-term
  to get new power series with the same R.

e. g. We can get a power series for $\ln(1-x)$ from the series for $\dfrac{1}{1-x}$

because we know $\dfrac{d}{dx}\ln(1-x) = \dfrac{-1}{1-x}$.

$$\ln(1-x) = -\int \frac{dx}{1-x} = -\int (1+x+x^2+\cdots)\,dx \qquad (R=1)$$

$$= -x - x^2/2 - x^3/3 - \cdots + C$$

We can evaluate C because we know $\ln(1)=0$.

Sub in $x=0$: $\quad 0 = \ln(1) = -0-0-\cdots+C$; so $C=0$

Answer: $\ln(1-x) = -x - x^2/2 - x^3/3 - \cdots$ with $R=1$.

Example

- Find a power series representation for $\dfrac{x^2}{(1+x)^2}$.

Start with $\dfrac{1}{1-x} = 1+x+x^2+\cdots$ for $|x|<1$.

Replace x by $-x$: $\dfrac{1}{1+x} = 1-x+x^2-x^3+\cdots$ for $|-x|<1$ or $|x|<1$ $(R=1)$

Differentiate Term-by-Term:

$-\dfrac{1}{(1+x)^2} = -1+2x-3x^2+4x^3-\cdots$ R still $=1$.

Multiply by $-x^2$:

$\dfrac{x^2}{(1+x)^2} = x^2-2x^3+3x^4-\cdots$ OR $\sum_{n=2}^{\infty}(-1)^n(n-1)x^n$ $(R=1)$

- Your answer should be a series arranged in increasing powers of x.

Evaluating Integrals Using Power Series

Some integrals that are difficult to evaluate with elementary antiderivatives can be evaluated with power series.

e. g. $\int \frac{dx}{1+x^3}$ — express $\frac{1}{1+x^3}$ as a power series, & integrate term-by-term

$\frac{1}{1-x} = 1 + x + x^2 + x^3 + \ldots$. Replace x by $(-x^3)$.

$\frac{1}{1+x^3} = 1 - x^3 + x^6 - x^9 + \ldots$ for $|-x^3| < 1$ or $|x| < 1$
$(R=1)$

$$\int \frac{dx}{1+x^3} = C + x - \frac{x^4}{4} + \frac{x^7}{7} - \frac{x^{10}}{10} + \ldots$$

C goes first — since it is the coefficient of x^0, & we want increasing powers of x.