Integration by Parts

- Integration using substitution is based on reversing the chain rule for differentiation.

- Integration by parts is based on reversing the product rule for differentiation:

$$d(uv) = u\,dv + v\,du$$

so— $\int u\,dv = \int d(uv) - \int v\,du$

now $\int d(uv) = uv + C$

$$\boxed{\int u\,dv = uv - \int v\,du}$$ MEMORIZE

- $\int u\,dv$ is converted To a new problem — find $\int v\,du$

old problem

Examples (1)– Integration by Parts

$$\int u\,dv = uv - \int v\,du$$

- $\int x \sin x\, dx$

Let $u = x \qquad dv = \sin x\, dx$

then $du = dx \qquad v = -\cos x$

$$= -x\cos x - -\int \cos x\, dx = -x\cos x + \sin x + C$$

- $\int \frac{\ln x}{x^2}\, dx$

Let $u = \ln x \qquad dv = dx/x^2$

then $du = dx/x \qquad v = -1/x$

$$= -\frac{\ln x}{x} - -\int \frac{dx}{x^2} = -\frac{\ln x}{x} - \frac{1}{x} + C$$

Remember — check answer by differentiating

Examples (2)– Integration by Parts

$$\int u\,dv = uv - \int v\,du$$

- $\int x^2 e^{5x}\,dx$

$$u = x^2 \qquad dv = e^{5x}dx$$
$$du = 2x\,dx \qquad v = \frac{e^{5x}}{5}$$

$$= \frac{x^2}{5}e^{5x} - \frac{2}{5}\int x e^{5x}dx$$ ← simpler than original $\int$ do parts again

$$u = x \qquad dv = e^{5x}dx$$
$$du = dx \qquad v = \frac{e^{5x}}{5}$$

$$= \frac{x^2}{5}e^{5x} - \frac{2}{5}\left[\frac{x}{5}e^{5x} - \frac{1}{5}\int e^{5x}dx\right]$$

$$= \frac{x^2}{5}e^{5x} - \frac{2x}{25}e^{5x} + \frac{2}{125}e^{5x} + C$$

Examples (3)– Integration by Parts

- $\int e^x \cos x \, dx = e^x \sin x - \int e^x \sin x \, dx$

$u = e^x \quad dv = \cos x \, dx$

$du = e^x dx \quad v = \sin x$

just as hard as original $\int$
do parts again.

$u = e^x \quad dv = \sin x \, dx$

$du = e^x dx \quad v = -\cos x$

$= e^x \sin x - \left[-e^x \cos x - - \int e^x \cos x \, dx \right]$

original $\int$

Set $\boxed{X}$ = original $\int$

We have $\quad \boxed{X} = e^x \sin x + e^x \cos x - \boxed{X}$

Solve for $\boxed{X}$: $\quad \boxed{X} = \frac{1}{2}\left[e^x \sin x + e^x \cos x \right] + C$

- Don't forget C. $\quad$ You Try $\int \cos(\ln x) \, dx$.

Example – Definite Integral by Parts

$$\int_a^b u\,dv = \Big[uv\Big]_a^b - \int_a^b v\,du$$

- $\int_0^1 xe^{-x}dx \quad = \Big[-xe^{-x}\Big]_0^1 - - \int_0^1 e^{-x}dx$

$u = x \qquad dv = e^{-x}dx$

$du = dx \qquad v = -e^{-x}$

$$= \Big[-xe^{-x} - e^{-x}\Big]_0^1$$

$$= \big[-e^{-1} - e^{-1}\big] - \big[0 - 1\big] = 1 - 2/e$$

Example 2 – Definite Integral by Parts

- $\int_0^{\sqrt{2}/2} \sin^{-1} x\,dx \quad = \left[x\sin^{-1}x\right]_0^{\sqrt{2}/2} - \int_0^{\sqrt{2}/2} \frac{x\,dx}{\sqrt{1-x^2}}$

$u = \sin^{-1}x \qquad dv = dx$

$du = \frac{dx}{\sqrt{1-x^2}} \qquad v = x$

use substitution $s = 1-x^2$

(a) $\left[x\sin^{-1}x\right]_0^{\sqrt{2}/2} = \left(\frac{\sqrt{2}}{2}\right)\left(\frac{\pi}{4}\right) - 0 = \frac{\sqrt{2}\pi}{8}$

(b) $\int_0^{\sqrt{2}/2} \frac{x\,dx}{\sqrt{1-x^2}} = -\int_1^{1/2} \frac{1}{2}\frac{ds}{\sqrt{s}} = \left[-\sqrt{s}\right]_1^{1/2} = -\frac{1}{\sqrt{2}} + 1$

$s = 1-x^2 \qquad$ when $x=0,\ s=1$

$ds = -2\,\underline{x\,dx} \qquad$ when $x = \sqrt{2}/2,\ s = 1/2$

Answer: (a) − (b) $= \frac{\sqrt{2}\pi}{8} + \frac{1}{\sqrt{2}} - 1$