Integrals of Sines and Cosines (1)

- Trig identities you need to know:

$$\sin^2\theta + \cos^2\theta = 1$$

$$\sin^2\theta = \frac{1}{2}(1-\cos 2\theta)$$

$$\cos^2\theta = \frac{1}{2}(1+\cos 2\theta)$$

half-angle identities
use to $\int$ <u>even powers</u>
of sines & cosines

- e. g. $\int \cos^2 4x\,dx \;=\; \frac{1}{2}\int (1+\cos 8x)\,dx$

$$= \frac{x}{2} + \frac{\sin 8x}{16} + C$$

Integrals of Sines and Cosines (2)

- e. g. $\int \cos^2 x \sin^2 x \, dx = \frac{1}{4}\int (1+\cos 2x)(1-\cos 2x)\,dx$

$$= \frac{1}{4}\int (1-\cos^2 2x)\,dx$$

apply ½-angle identity again

$$= \frac{x}{4} - \frac{1}{8}\int (1+\cos 4x)\,dx$$

$$= \frac{x}{4} - \frac{x}{8} - \frac{\sin 4x}{32} + C$$

$$= \frac{x}{8} - \frac{\sin 4x}{32} + C$$

- These answers are trickier to check by differentiation. You need to use trig identities to get to the original form.

Integrals of Sines and Cosines (3)

- e. g. $\int \cos^3 x \sin^2 x \, dx$

  ↑ odd power – convert all but 1 cosine to sines. Use $\cos^2 x = 1 - \sin^2 x$

$$= \int (1 - \sin^2 x) \sin^2 x \underbrace{\cos x \, dx}_{d \sin x}$$

let $u = \sin x$

$du = \cos x \, dx$

$$= \int (1 - u^2) u^2 \, du = \int (u^2 - u^4) \, du$$

$$= \frac{u^3}{3} - \frac{u^5}{5} + C = \frac{\sin^3 x}{3} - \frac{\sin^5 x}{5} + C$$

- Use this idea for $\int \cos^m x \sin^n x \, dx$ when either m or n is odd.

Integrals with Tan, Sec, etc. (1)

- Integrals involving tan and sec, or cot and csc are attacked using the following facts:

$$\int \tan x\,dx = \int \frac{\sin x}{\cos x}\,dx = -\ln|\cos x| + C$$

$$\int \cot x\,dx = \int \frac{\cos x}{\sin x}\,dx = \ln|\sin x| + C$$

$$\int \sec^2 x\,dx = \tan x + C; \qquad \int \csc^2 x\,dx = -\cot x + C$$

$$\int \sec x \tan x\,dx = \sec x + C$$

$$\int \csc x \cot x\,dx = -\csc x + C$$

Integrals with Tan, Sec, etc. (2)

— more useful facts

- you know $\sin^2\theta + \cos^2\theta = 1$

divide by $\sin^2\theta$: $1 + \cot^2\theta = \csc^2\theta$

divide by $\cos^2\theta$: $\tan^2\theta + 1 = \sec^2\theta$

- $$\int \sec x \, dx = \int \sec x \left(\frac{\sec x + \tan x}{\sec x + \tan x}\right) dx$$

$$= \ln|\sec x + \tan x| + C$$

$$\int \csc x \, dx = \int \csc x \left(\frac{\csc x + \cot x}{\csc x + \cot x}\right) dx$$

$$= -\ln|\csc x + \cot x| + C$$

Integrals with Tan, Sec, etc. (3)

e. g. $\int \tan^4 x\,dx = \int \text{Tan}^2 x\,(\sec^2 x - 1)\,dx$

$= \int \text{Tan}^2 x \underbrace{\sec^2 x\,dx}_{d\,\text{Tan}\,x} - \int \text{Tan}^2 x\,dx$

$\sec^2 x - 1$

$= \dfrac{\text{Tan}^3 x}{3} - \text{Tan}\,x + x + C$

e. g. $\int \sec^3 x \tan x\,dx = \int \sec^2 x\,(\sec x\,\text{Tan}\,x\,dx)$

$d\sec x$

$= \dfrac{\sec^3 x}{3} + C$