Trig Substitutions

- Useful for integrands involving $\sqrt{a^2-x^2}$, $\sqrt{a^2+x^2}$, $\sqrt{x^2-a^2}$ (also for 3/2 powers)

When integrand contains	Substitute	Then:
$\sqrt{a^2-x^2}$	$x = a\sin\theta$	$\sqrt{a^2-x^2}$ becomes $a\cos\theta$
$\sqrt{a^2+x^2}$	$x = a\tan\theta$	$\sqrt{a^2+x^2}$ becomes $a\sec\theta$
$\sqrt{x^2-a^2}$	$x = a\sec\theta$	$\sqrt{x^2-a^2}$ becomes $a\tan\theta$

- Note: $a > 0$

Example 1 – Sine Substitution

- $\displaystyle\int \frac{\sqrt{1-x^2}}{x^2}\,dx \quad = \int \frac{\cos\theta}{\sin^2\theta}\cos\theta\,d\theta = \int \cot^2\theta\,d\theta$

$(a=1)$ $\quad x = \sin\theta$ $\qquad\qquad\qquad\qquad = \displaystyle\int (\csc^2\theta - 1)\,d\theta$

so $1-x^2 = 1-\sin^2\theta = \cos^2\theta$

& $\quad dx = \cos\theta\,d\theta$ $\qquad\qquad\qquad\qquad = -\cot\theta - \theta + C$

We must answer in terms of x:

$\theta = \sin^{-1}x$

$= \sqrt{1-x^2}$

from the △,

$$\cot\theta = \frac{\sqrt{1-x^2}}{x}$$

Answer: $\displaystyle -\frac{\sqrt{1-x^2}}{x} - \sin^{-1}x + C$

[check by differentiating]

Example 2 – Secant Substitution

- $$\int \frac{dx}{\sqrt{x^2-16}} = \int \frac{4\sec\theta \tan\theta \, d\theta}{4\tan\theta} = \int \sec\theta \, d\theta$$

$(a=4)$ $\quad x = 4\sec\theta$

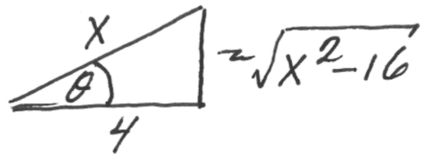
so $\quad x^2 - 16 = 16(\sec^2\theta - 1) = 16\tan^2\theta$

& $\quad dx = 4\sec\theta \tan\theta \, d\theta$

$$= \int \sec\theta \left(\frac{\sec\theta + \tan\theta}{\sec\theta + \tan\theta} \right) d\theta = \ln|\sec\theta + \tan\theta| + C$$

Answer in Terms of x:

$\sec\theta = x/4$



$\tan\theta = \dfrac{\sqrt{x^2-16}}{4}$

Answer: $\ln\left| \dfrac{x}{4} + \dfrac{\sqrt{x^2-16}}{4} \right| + C$ OR $\ln\left|x + \sqrt{x^2-16}\right| + C$

Example 3 – Tangent Substitution

- $\displaystyle\int \frac{dx}{(25+4x^2)^{3/2}} = \frac{1}{2}\int \frac{du}{(25+u^2)^{3/2}} = \frac{1}{2}\int \frac{5\sec^2\theta\, d\theta}{(25\sec^2\theta)^{3/2}}$

$u = 2x$

$du = 2\,dx$

$(a=5)$ $u = 5\tan\theta$

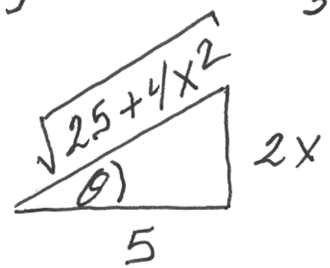
$25+u^2 = 25(1+\tan^2\theta) = 25\sec^2\theta$

$du = 5\sec^2\theta\, d\theta$

$$= \frac{1}{2}\int \frac{d\theta}{5^2\sec\theta} = \frac{1}{50}\int \cos\theta\, d\theta = \frac{1}{50}\sin\theta + C$$

Answer in Terms of x:

$\tan\theta = \frac{u}{5} = \frac{2x}{5}$



Answer: $\displaystyle\frac{1}{50}\frac{2x}{\sqrt{25+4x^2}} + C$ OR $\displaystyle\frac{x}{25\sqrt{25+4x^2}} + C$