Integrating Rational Functions – 1

- A rational function is a ratio of polynomials $P(x)/Q(x)$
- To integrate a rational function, ① first check whether degree $P <$ degree Q. If not, use long division to get

$$P(x)/Q(x) = p(x) + R(x)/Q(x)$$

where p is a polynomial & degree $R <$ degree Q.

- e.g. $\int \frac{x^3}{x-1}\,dx = \int \left(x^2 + x + 1 + \frac{1}{x-1}\right) dx = \ldots$

(you finish this)

- For the rest of this lecture we'll assume that we have $\int \frac{P(x)}{Q(x)}\,dx$ with degree $P <$ degree Q.

Integrating Rational Functions – 2

- ② Next see if $\frac{P}{Q}$ is a known derivative or if a simple substitution will help.

- ③ Otherwise, you can use partial fractions to rewrite $\frac{P}{Q}$ as a sum of simpler functions.

Partial Fraction Expansions – 1

- Completely factor $Q(x)$ into linear & irreducible quadratic factors. Use only real #'s.
- If you have n factors, you will rewrite $\frac{P}{Q}$ as a sum of n simpler functions.
- e. g. $\frac{1}{x^3+2x} = \frac{1}{x(x^2+2)} = \frac{A}{x} + \frac{Bx+C}{x^2+2}$

where A, B & C are uniquely determined constants.

- The numerator is constant if the denominator is based on a linear factor. e.g. A/x
- The numerator is linear if the denominator is based on a quadratic factor. e.g. $(Bx+C)/(x^2+2)$

Partial Fraction Expansions – 2

- e. g. $$\frac{1}{(x+1)(3x+2)} = \frac{A}{x+1} + \frac{B}{3x+2}$$

- e. g. $$\frac{x}{(x+2)^3} = \frac{A}{x+2} + \frac{B}{(x+2)^2} + \frac{C}{(x+2)^3}$$

  3 factors $\Rightarrow$ 3 Terms

- e. g. $$\frac{x^2+4}{(x^2+1)^2} = \frac{Ax+B}{x^2+1} + \frac{Cx+D}{(x^2+1)^2}$$

  2 factors $\Rightarrow$ 2 Terms

Next we'll learn how to find $A, B, C, \ldots$

Partial Fractions Example 1

- $\int \frac{dx}{x^2+3x+2} = \int \frac{dx}{(x+1)(x+2)}$

$$\frac{1}{(x+1)(x+2)} = \frac{A}{x+1} + \frac{B}{x+2}$$

$$1 = A(x+2) + B(x+1) \quad \text{for all } x$$

$$1 = (A+B)x + (2A+B)$$

(polynom = polynom implies <u>equal coeffs</u>. of corresponding x-powers)

$$\left.\begin{aligned} A+B&=0 \\ 2A+B&=1 \end{aligned}\right\} \quad \text{solve: } A=1 \text{ \& } B=-1$$

Answer: $\int \left(\frac{1}{x+1} - \frac{1}{x+2}\right) dx = \ln\left|\frac{x+1}{x+2}\right| + C$

Partial Fractions Example 2

- $\int \frac{x+1}{x(x^2+1)}\,dx$

$$\frac{x+1}{x(x^2+1)} = \frac{A}{x} + \frac{Bx+C}{x^2+1}$$

$$x+1 = A(x^2+1) + (Bx+C)x$$

$$x+1 = (A+B)x^2 + Cx + A$$

$$\left.\begin{aligned} A+B &= 0 \\ C &= 1 \\ A &= 1 \end{aligned}\right\} \quad A=1,\ B=-1,\ C=1$$

Answer: $\int \left(\frac{1}{x} + \frac{(-x+1)}{x^2+1}\right) dx$

$$= \int \left(\frac{1}{x} - \frac{x}{x^2+1} + \frac{1}{x^2+1}\right) dx = \ln|x| - \frac{1}{2}\ln(x^2+1) + \mathrm{Tan}^{-1}x + C$$

Partial Fractions Example 3

- A quick way to get the constants when you have distinct linear factors of $Q(x)$:

$$\frac{2x+3}{x(x+1)(x-2)} = \frac{A}{x} + \frac{B}{x+1} + \frac{C}{x-2}$$

$$2x+3 = A(x+1)(x-2) + Bx(x-2) + Cx(x+1)$$

zeros of Q are $0, -1, 2$

sub in $x=0$: $\quad 3 = A(1)(-2) + 0 + 0$, $\quad$ so $A = -3/2$

" $\quad x=-1$: $\quad 1 = 0 + B(-1)(-3) + 0$, $\quad$ so $B = 1/3$

" $\quad x=2$: $\quad 7 = 0 + 0 + C(2)(3)$, $\quad$ so $C = 7/6$

$$\int \frac{2x+3}{x(x+1)(x-2)}\,dx = -\tfrac{3}{2}\ln|x| + \tfrac{1}{3}\ln|x+1| + \tfrac{7}{6}\ln|x-2| + C$$