Approximate (Numerical) Integration

For approximating the value of a definite integral $\int_a^b f(x)\,dx$

- used when cannot find an elementary antiderivative of f(x)

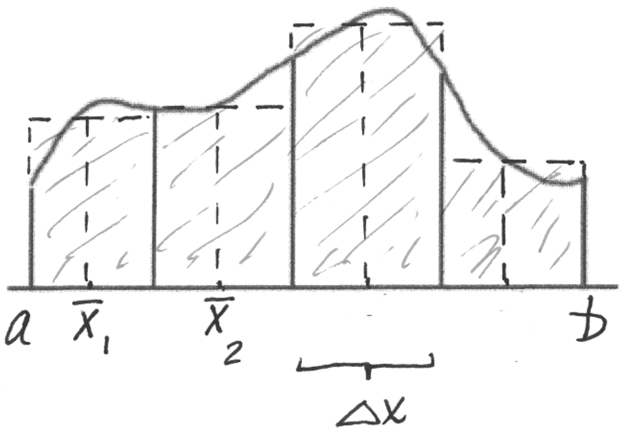
$$\text{e.g.} \int_0^1 e^{x^2}\,dx \qquad \text{or} \qquad \int_0^5 \frac{dx}{\sqrt{1+x^3}}$$

- or when we only have a list of f-values (e.g. from data acquisition)

In section 5.2 we learned to estimate definite integrals (areas under graphs) with a Midpoint Rule (review 5.2)

Today we'll look at 3 different rules for numerical integration: the Midpoint Rule, the Trapezoidal Rule and Simpson's Rule.

The Midpoint Rule



A method to approx. $\int_a^b f(x)\,dx$

Partition $[a, b]$ into n equal subintervals of length $\boxed{\Delta x = \frac{(b-a)}{n}}$

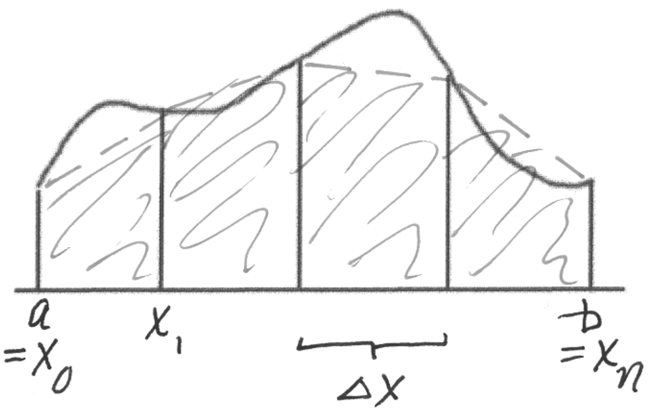
Let $\bar{y}_i = f(\bar{x}_i)$

Sum of areas of rectangles is $\Delta x(\bar{y}_1 + \bar{y}_2 + \dots \bar{y}_n)$

The midpoint approximation is:

$$\int_a^b f(x)\,dx \approx M_n = \Delta x(\bar{y}_1 + \bar{y}_2 + \dots + \bar{y}_n)$$

The Trapezoidal Rule



Partition $[a, b]$ into n equal subintervals: $\boxed{\Delta x = \dfrac{(b-a)}{n}}$

$y_i = f(x_i)$

Area of Trapezoid

is $(\text{base})\begin{pmatrix}\text{aver.}\\ \text{height}\end{pmatrix} = \Delta x\left(\dfrac{y_1 + y_2}{2}\right)$

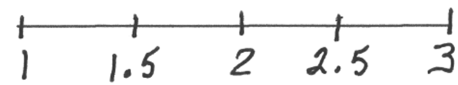
Sum of Trapezoids: $\dfrac{\Delta x}{2}\left[(y_0 + y_1) + (y_1 + y_2) + \dots + (y_{n-1} + y_n)\right]$

<u>Trapezoidal approximation</u>: $\displaystyle\int_a^b f(x)\,dx \approx T_n = \frac{\Delta x}{2}\left(y_0 + 2y_1 + 2y_2 + \dots + 2y_{n-1} + y_n\right)$

Example – Trapezoidal Rule

Use the Trapezoidal Rule to approximate $\int_1^3 e^{-x^2}\,dx$ with $n = 4$ subdivisions.

$$\Delta x = \frac{(3-1)}{2} = \frac{1}{2}$$

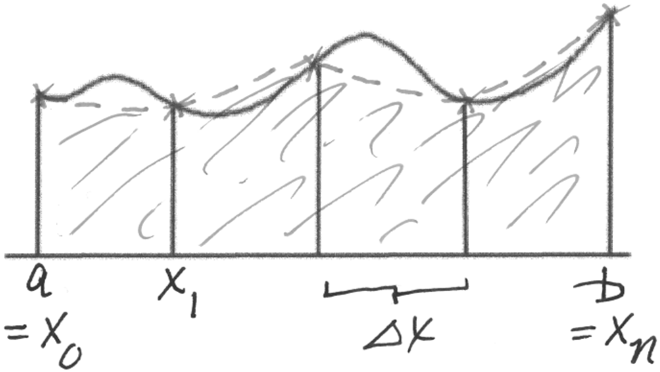


$$\int_1^3 e^{-x^2}\,dx \approx T_4 = \frac{(1/2)}{2}\left(e^{-1} + 2e^{-(1.5)^2} + 2e^{-4} + 2e^{-(2.5)^2} + e^{-9}\right)$$

$$= .15482$$

- If we do $n = 8$, T_8 is $.14322$
- Actual value (to 5 places) is $\int_1^3 e^{-x^2}\,dx = .13938$

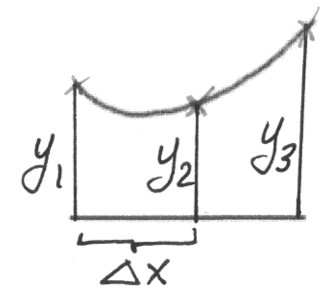
Simpson's Rule



Uses parabolic approximation.

$\Delta x = \frac{(b-a)}{n}$. $\quad$ n must be even.

area under parabola is $\frac{\Delta x}{3}(y_1 + 4y_2 + y_3)$



Simpson's Rule approximation:

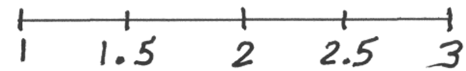
$$\int_a^b f(x)\,dx \approx S_n = \frac{\Delta x}{3}\left(y_0 + 4y_1 + 2y_2 + 4y_3 + \ldots + 4y_{n-1} + y_n\right)$$

- note pattern of coeffs $\quad$ 1–4–2–4–2–...–4–2–4–1

Example – Simpson's Rule

Use Simpson's Rule to approximate $\int_1^3 e^{-x^2}\,dx$ with $n = 4$ subdivisions.

$$\Delta x = \frac{(3-1)}{4} = \frac{1}{2}$$



$$\int_1^3 e^{-x^2}\,dx$$

$$\approx S_4 = \frac{(1/2)}{3}\left(e^{-1} + 4e^{-(1.5)^2} + 2e^{-4} + 4e^{-(2.5)^2} + e^{-9}\right)$$

$$= .13899$$

- If we do $n = 8$, S_8 is $.13935$
- Actual value (to 5 places) is $\int_1^3 e^{-x^2}\,dx = .13938$

Comparison of Rules

For the integral $\int_1^3 e^{-x^2}\,dx$

Midpoint Rule	Trapezoidal Rule	Simpson's Rule
$M_4 = .13162$	$T_4 = .15482$	$S_4 = .13899$
$M_8 = .13746$	$T_8 = .14322$	$S_8 = .13935$

Actual Value (to 5 places) : .13938

For the same amount of computational effort, the Midpoint Rule is better than the Trapezoidal Rule, and Simpson's Rule is best.