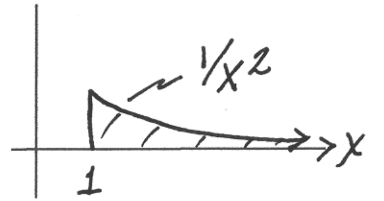
Improper Integrals - 1

- $\int_a^b f(x)\,dx$ is an improper integral if $b = \infty$ or $a = -\infty$ or if f has an infinite discontinuity somewhere in $[a,b]$.

- We must evaluate all improper integrals using limits.

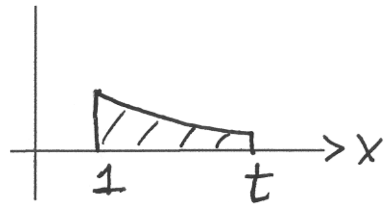
- e. g. $\int_1^\infty \frac{dx}{x^2}$ could reasonably be interpreted as the shaded area, provided this area is finite.



¿is area finite?

We define

$$\int_1^\infty \frac{dx}{x^2} = \lim_{t\to\infty} \int_1^t \frac{dx}{x^2}$$



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Improper Integrals – 2

- $\int_1^\infty \frac{dx}{x^2} = \lim_{t\to\infty} \int_1^t \frac{dx}{x^2} = \lim_{t\to\infty} \left[-\frac{1}{x}\right]_1^t$

  $= \lim_{t\to\infty} \left[-\frac{1}{t} - -1\right] = 1$

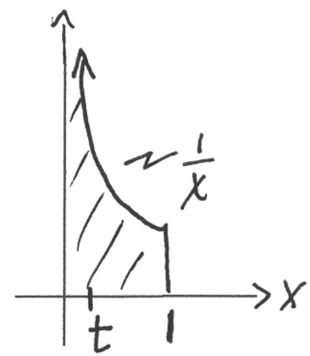
- We say an improper integral converges when it has a finite value. Otherwise, it diverges.

- e. g. $\int_0^1 \frac{dx}{x} = \lim_{t\to 0^+} \int_t^1 \frac{dx}{x}$

  $\frac{1}{x}$ undefined at 0

  $= \lim_{t\to 0^+} [\underbrace{\ln 1}_{=0} - \underbrace{\ln t}_{\to -\infty}]$

  $= +\infty$



This integral diverges (To $+\infty$)

Improper Integrals – 3

- The example $\int_1^\infty \frac{dx}{x^2}$ converges because $\frac{1}{x^2}$ "squeezes down" against the x-axis quickly enough to trap a finite area as $x \to \infty$.

- The example $\int_0^1 \frac{dx}{x}$ diverges because $\frac{1}{x}$ does not "squeeze down" against the y-axis quickly enough to trap a finite area as $x \to 0$.

Setting Up Improper Integrals – 1

- First identify the values that make the integral improper. (I'll call them improper values.)

- Break the integral into a sum of integrals, each having exactly one improper value as an endpoint.

- e. g. $\int_{-1}^{1} \frac{dx}{x^2} = \int_{-1}^{0} \frac{dx}{x^2} + \int_{0}^{1} \frac{dx}{x^2}$

improper value: 0

$$= \lim_{t \to 0^-} \int_{-1}^{t} \frac{dx}{x^2} + \lim_{s \to 0^+} \int_{s}^{1} \frac{dy}{s^2}$$

... you do this ...

$$= \infty + \infty = \infty$$

Integral diverges.

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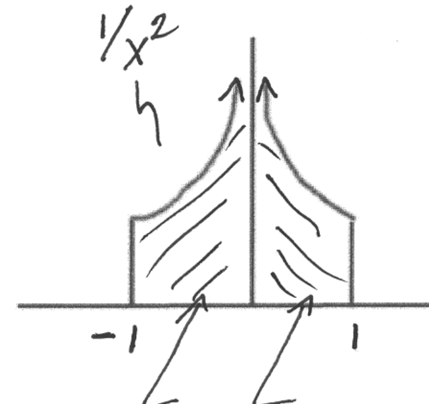
Setting Up Improper Integrals – 2

We just showed that $\displaystyle\int_{-1}^{1} \frac{dx}{x^2}$ diverges (to $+\infty$).



neither area is finite

If we tried to calculate the integral as

$$\left[-\frac{1}{x}\right]_{-1}^{1} = (-1)-(1) = -2$$

we would clearly get a ridiculous answer for the area under $1/x^2$ between -1 & 1.

— so using limits to calculate improper $\int$'s is necessary —

Examples – Improper Integrals

- $\int_{0}^{\infty} \frac{dx}{x^2} = \underbrace{\lim_{t \to 0^+} \int_t^1 \frac{dx}{x^2}}_{\text{diverges (to } \infty)} + \underbrace{\lim_{s \to \infty} \int_1^s \frac{dx}{x^2}}_{\text{converges to 1}}$

improper values: 0 & ∞

so the whole integral diverges.

- $\int_{0}^{2} \frac{dx}{\sqrt{2-x}} = \lim_{t \to 2^-} \int_0^t \frac{dx}{\sqrt{2-x}} = \lim_{t \to 2^-} \left[-2\sqrt{2-x}\right]_0^t$

improper value: 2

$$= \lim_{t \to 2^-} \left[-2\sqrt{2-t} + 2\sqrt{2}\right]$$

$$= 0 + 2\sqrt{2} = 2\sqrt{2}$$

Examples – Improper Integrals

- $\int_0^\infty \sin 5x \, dx = \lim_{t\to\infty} \int_0^t \sin 5x \, dx = \lim_{t\to\infty} \left[-\frac{1}{5}\cos 5x\right]_0^t$

improper
value: ∞

$$= \lim_{t\to\infty} \left[\underbrace{-\frac{1}{5}\cos 5t}_{\text{oscillates}} + \frac{1}{5}\right]$$

limit does not exist—
so the integral diverges