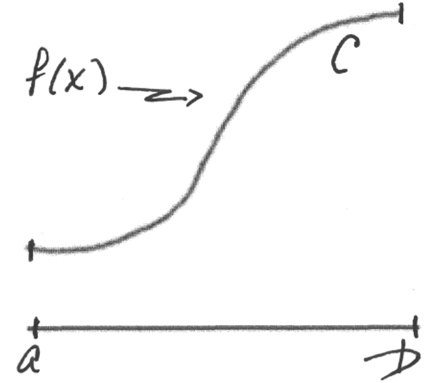
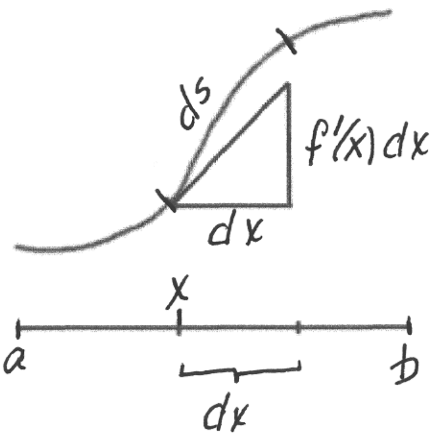
Length of a Smooth Curve

A smooth curve C is defined by $y = f(x)$ for $a \le x \le b$ where $f(x)$ and $f'(x)$ are continuous.

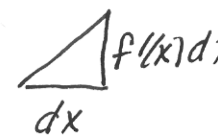
? Length of C ?



Subdivide by thin vertical slices:



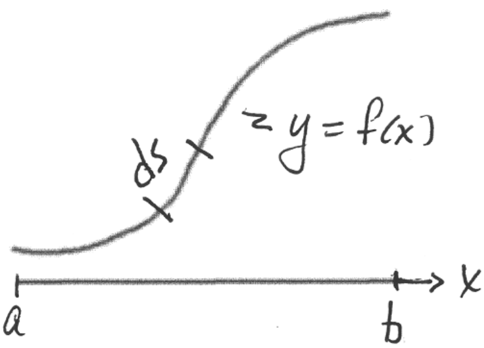
Arc length of slice ≈ hypotenuse of



$$ds = \sqrt{(dx)^2 + [f'(x)\,dx]^2}$$

$$\boxed{ds = \sqrt{1 + [f'(x)]^2}\,dx} \quad \& \quad L_{\text{ength}} = \int_{x=a}^{x=b} ds$$

Arc Length Formulas



differential of arc length = ds

$$ds = \sqrt{1+[f'(x)]^2}\,dx$$

$$\& \quad L_{ength} = \int_{x=a}^{x=b} ds$$

If the curve is given as $x = g(y)$, for $y \in [c, d]$

$$ds = \sqrt{1+[g'(y)]^2}\,dy$$

$$\& \quad L = \int_{y=c}^{y=d} ds$$

- Simplify ds as much as possible before trying to integrate.

Example – Arc Length

Find the length of the curve $y = 2x^{3/2}$ for $1 \le x \le 3$.

- A picture of the actual curve is not necessary.

$$ds = \sqrt{1 + [f'(x)]^2}\, dx \quad \text{with } f(x) = 2x^{3/2}$$

$$f'(x) = 3x^{1/2}$$

$$ds = \sqrt{1 + 9x}\, dx$$

$$\text{Length} = \int_1^3 \sqrt{1+9x}\, dx = \frac{2}{(3)(9)} \left[(1+9x)^{3/2} \right]_1^3$$

$$= \frac{2}{27} \left[28^{3/2} - 10^{3/2} \right].$$

Example – Arc Length

Find the length of the curve $x = \frac{1}{8}y^2 - \ln y$ for $1 \le y \le 2$.

$$ds = \sqrt{1 + [g'(y)]^2}\, dy \quad \text{with } g(y) = \tfrac{1}{8}y^2 - \ln y$$

$$g'(y) = \tfrac{1}{4}y - \tfrac{1}{y}$$

$$ds = \sqrt{1 + \tfrac{1}{16}y^2 - \tfrac{1}{2} + \tfrac{1}{y^2}}\, dy$$

$$= \sqrt{\tfrac{1}{16}y^2 + \tfrac{1}{2} + \tfrac{1}{y^2}}\, dy = \sqrt{\left(\tfrac{1}{4}y + \tfrac{1}{y}\right)^2}\, dy = \left(\tfrac{1}{4}y + \tfrac{1}{y}\right) dy$$

$$\text{Length} = \int_1^2 \left(\tfrac{1}{4}y + \tfrac{1}{y}\right) dy = \left[\tfrac{1}{8}y^2 + \ln|y|\right]_1^2$$

$$= \tfrac{1}{2} + \ln 2 - \tfrac{1}{8} - 0 = \tfrac{3}{8} + \ln 2$$