Natural (Exponential) Growth

Natural growth problems occur when a population grows (or decays) at a rate proportional to the size (P) of the population:

$$\frac{dP}{dt} = kP \qquad (t \text{ is Time, } k \text{ is constant})$$

$\dfrac{dP}{dt}$ is the growth rate

k is the relative growth rate: $k = \dfrac{(dP/dt)}{P}$

$k > 0$: growth
$k < 0$: decay

Applications:
- bacterial growth
- radioactive decay
- chemical concentration during reaction
- continuous compounding of interest

Solution of Natural Growth Differential Equation

- assume $\frac{dy}{dt} = ky$ and $y(0) = y_0$ (k constant)

$$\frac{dy}{y} = k\,dt$$

$$\int \frac{dy}{y} = k \int dt$$

$$\ln|y| = kt + C$$

$$y = \pm e^{kt+C} = \pm e^{C} e^{kt}$$

sub in $t=0$ to find $\pm e^{C}$: $\quad y(0) = y_0 = \pm e^{C}(1)$

- solution $\boxed{y(t) = y_0 e^{kt}}$

(This says a population with constant relative growth rate (k) grows exponentially)

Example – Bacteria Growth

A bacteria culture starts with 200 bacteria and grows at a constant relative rate. After 5 hours there are 7,000 bacteria.

(a) Find the relative growth rate

(b) When will the population reach 18,000?

We know $P = P_0 e^{kt}$

$P = 200 e^{kt}$ (given $P_0 = 200$)

$7{,}000 = 200 e^{5k}$ (given $P = 7{,}000$ when $t = 5$)

solve for k: $e^{5k} = \frac{7{,}000}{200} = 35$

$5k = \ln 35$

Answer (a) $k = (\ln 35)/5 \approx .7107$

For (b), we want t when $18{,}000 = 200 e^{(t \ln 35)/5}$

$\ln 90 = (t \ln 35)/5$

Answer (b) $t = \frac{5 \ln 90}{\ln 35} \approx 6.328$ hours.

Example – Radioactive Decay

The half-life of a radioactive sample is the time required for half the sample to decay.

- The half-life of radium-226 is 1590 years.

(a) If a sample has mass m of 250 mg, find the mass remaining after 10 years.

(b) When will the mass be reduced to 100 mg?

we know $m = m_0 e^{kt} = 250 e^{kt}$

and $\frac{250}{2} = 250 e^{1590k}$ (half mass remains when $t = 1590$)

we can solve this for k:

$$1590k = \ln(1/2) = -\ln 2$$

$$k = -\frac{\ln 2}{1590} \approx -.000436$$

Example – Radioactive Decay (concluded)

from previous slide: $m = 250e^{kt}$

$$k = -\frac{\ln 2}{1590}$$

(a) mass remaining after 10 years:

$$m = 250\, e^{-(10 \ln 2)/1590} \approx 239.336 \text{ mg.}$$

(b) when is $m = 100$ mg?

We must solve for t: $100 = 250\, e^{-(t \ln 2)/1590}$

$$\ln(10/25) = -(t \ln 2)/1590$$

$$t = -\frac{1590 \ln(10/25)}{\ln 2}$$

$$\approx 2{,}101.866 \text{ years.}$$

- make sure the answers to (a) & (b) are reasonable.