Sequences

- A sequence is an ordered, unending list of real numbers:
  \[ a_1, a_2, a_3, \ldots, a_n, \ldots \]
  Alternative notation: \( \{a_n\} \) or \( \{a_n\}_{n=1}^{\infty} \).
  \( a_n \) is called the \( n^{\text{th}} \) Term.

- It is just a special type of function, with domain restricted to positive integers.

- Principal question about a sequence: Does it converge as \( n \to \infty \), and if so, what is the limit? (i.e. does graph have horizontal asymptote?)
Finding Sequence Limits

To find the limit of \( \{a_n\} \):

- we can use the limit laws that we learned for functions, including the Squeeze Theorem (section 2.3)
- and techniques for horizontal asymptotes (section 2.6)
- if \( a_n = f(n) \), where the function \( f(x) \) is defined for large positive \( x \), then l’Hospital’s Rule may be useful

A useful fact: \[ \lim_{n \to \infty} |a_n| = 0, \text{ then } \lim_{n \to \infty} a_n = 0. \]

**IMPORTANT** - The limit of a convergent sequence must be finite.
Sequence Examples – 1

List the first 4 terms of each sequence \( \{a_n\} \). Does \( \{a_n\} \) converge or diverge? If it converges, find the limit.

• \( \{a_n\} = \left\{ \frac{n}{n+1} \right\} = \left\{ \frac{1}{2}, \frac{2}{3}, \frac{3}{4}, \frac{4}{5}, \ldots \right\} \) (looks like \( \lim_{n \to \infty} a_n = 1 \))

\[
\lim_{n \to \infty} \frac{n}{n+1} = \lim_{n \to \infty} \frac{1}{1+\frac{1}{n}} = \frac{1}{1+0} = 1
\]

answer: \( \{a_n\} \) converges to 1.

• \( \{a_n\} = \left\{ \frac{e^n}{n^2} \right\} = \left\{ e, \frac{e^2}{4}, \frac{e^3}{9}, \frac{e^4}{16}, \ldots \right\} \) (difficult to predict limit)

\[
\lim_{x \to \infty} \frac{e^x}{x^2} \overset{\text{L'Hopital}}{=} \lim_{x \to \infty} \frac{e^x}{2x} \overset{\text{L'Hopital}}{=} \lim_{x \to \infty} \frac{e^x}{2} = \infty
\]

answer: \( \{a_n\} \) diverges. (To \( \infty \)
Sequence Examples – 2

• \( \{a_n\} = \{n \cos n\pi\} = \{\cos \pi, 2 \cos 2\pi, 3 \cos 3\pi, 4 \cos 4\pi, \ldots\} \)
  \[= \{-1, +2, -3, +4, \ldots\}\]
  pattern shows no finite limit as \(n \to \infty\).

  \[\text{answer: } \{a_n\} \text{ diverges.}\]

• \( \{a_n\} = \left\{\frac{(-1)^n}{n+2}\right\} = \left\{-\frac{1}{3}, +\frac{1}{4}, -\frac{1}{5}, +\frac{1}{6}, \ldots\right\}\) (looks like \(\lim_{n \to \infty} an = 0\))

  \[\lim_{n \to \infty} |a_n| = \lim_{n \to \infty} \frac{1}{n+2} = \frac{1}{\infty} = 0\]

  answer: \(\{a_n\}\) converges to \(0\).