Alternating Series

- are series whose terms are alternately positive & negative.

  \[ \text{e.g. } 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \ldots \]

  
  Standard notation \( \sum (-1)^n b_n \) or \( \sum (-1)^{n+1} b_n \) or \( \sum (-1)^{n-1} b_n \)

  where \( b_n > 0 \) for all \( n \).

- The Alternating Series Test:

  If an alternating series satisfies

  1. \( \lim_{n \to \infty} b_n = 0 \)

  AND 2. \( b_{n+1} \leq b_n \) for all \( n \) (i.e. \( b_n \)'s decrease)

  then the series converges.
Convergence Picture and Remainder Estimate

\[\sum_{n=1}^{\infty} (-1)^{n+1} b_n = b_1 - b_2 + b_3 - \ldots\]

The \(n^{th}\) partial sum is \(S_n = b_1 - b_2 + \ldots \pm b_n\)

- The partial sums oscillate about the sum \(S\)

- Remainder estimate –

  \[R_n = S - S_n\] satisfies \(|R_n| \leq b_{n+1}\]
Examples – Alternating Series

• Determine whether \( \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n} \) converges or diverges.

Series is \( 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \cdots \) alternating series; \( b_n = \frac{1}{n} \)

Check hypotheses:

1. \( \lim_{n \to \infty} b_n = \lim_{n \to \infty} \frac{1}{n} = 0 \) \( \checkmark \)

of Alt. Series Test:

2. \( \frac{1}{n+1} \leq \frac{1}{n} \) (obvious) for all \( n \) \( \checkmark \)

Conclusion: \( \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n} \) converges by the Alternating Series Test.

• \( \sum_{n=2}^{\infty} \frac{(-1)^n}{\ln n} = \frac{1}{\ln 2} - \frac{1}{\ln 3} + \frac{1}{\ln 4} - \cdots \) alternating series; \( b_n = \frac{1}{\ln n} \)

1. \( \lim_{n \to \infty} \frac{1}{\ln n} = 0 \) \( \checkmark \)

(2) \( \frac{1}{\ln(n+1)} \leq \frac{1}{\ln n} \) since \( \ln(n+1) > \ln n \) \( \checkmark \)

Conclusion: The series converges by the Alt. Series Test.
Example – Alternating Series

Note - the condition \( b_{n+1} \leq b_n \) for all \( n \) can be replaced by \( b_{n+1} \leq b_n \) for all \( n \geq N \) where \( N \) is some fixed number.

\[
\sum_{n=1}^{\infty} (-1)^{n+1} \frac{n^2}{n^3 + 1} = \frac{1}{2} - \frac{4}{9} + \frac{9}{28} - \frac{16}{65} + \ldots \quad \text{alternating series}
\]

\[
\begin{align*}
(1) \lim_{n \to \infty} \frac{n^2}{n^3 + 1} &= \lim_{n \to \infty} \frac{1/n}{1 + 1/n^3} = 0, \\
(2) \text{Do the } b_n \text{'s decrease? They will if } f(x) = \frac{x^2}{x^3 + 1} \text{ decreases.}
\end{align*}
\]

So we'll show \( f' < 0 \).

\[
f'(x) = \frac{2x(x^3 + 1) - x^2(3x^2)}{(x^3 + 1)^2} = \frac{2x - x^4}{(x^3 + 1)^2} = \frac{x(2-x^3)}{(x^3 + 1)^2}
\]

\[
= (+)(-) < 0
\]

\[
\text{Conclusion: The series converges by the Alt. Series Test.}
\]
Example – Remainder (Error) Estimate

- How many terms should we add in order to find the sum of
  \[ \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n^3} \] with error < 0.01?

Since \[ |R_n| \leq b_{n+1} = \frac{1}{(n+1)^3} \], we want the smallest integer \( n \) satisfying \( \frac{1}{(n+1)^3} < 0.01 \)

\[ (n+1)^3 > \frac{1}{0.01} = 100 \]
\[ n > (100)^{1/3} - 1 \approx 3.64 \]

Answer: \( n = 4 \) terms.

We have shown that
\[ \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n^3} \approx 1 - \frac{1}{2^3} + \frac{1}{3^3} - \frac{1}{4^3} = .896412 \]
with error < 0.01