Power Series

A power series has the form
\[ \sum_{n=0}^{\infty} c_n (x-a)^n = c_0 + c_1 (x-a) + c_2 (x-a)^2 + \ldots \]

- \( x \) is a variable
- \( a \) is a constant (often \( a = 0 \))
- The \( c_n \)'s are called the coefficients.

(A power series looks like an “unending polynomial” in powers of \( x-a \). True polynomials always end.)

1. For each power series, there is a number \( R \) \( (0 \leq R \leq \infty) \) so that \( \sum_{n=0}^{\infty} c_n (x-a)^n \) converges absolutely for \( |x-a| < R \) and diverges for \( |x-a| > R \).

2. \( R \) is the radius of convergence. We find \( R \) using the ratio or root test.
Interval of Convergence

- The principal question about a power series is: For which values of \( x \) does the series converge?

- The interval of convergence consists of all \( x \)-values for which the series converges. It includes \( a-R < x < a+R \)
  \( a \) may include \( a-R \) or \( a+R \) or both, depending on the specific power series.

\[
\text{Series converges absolutely}
\]

\[
\text{diverges}
\]

\[
\text{diverges}
\]

11.8 #2
Examples – Power Series -1

- Find R and the interval of convergence of \( \sum_{n=0}^{\infty} x^n = 1 + x + x^2 + \ldots \)

We already know all about this series – since it is a geometric series with \( r = x \).

So: The series converges for \( |x| < 1 \) and diverges for \( |x| > 1 \).

i.e. \( \sum_{n=0}^{\infty} x^n \) has interval of convergence \( -1 < x < 1 \).

\[
\begin{array}{c}
\text{diverges} \\
-1 \quad 0 \quad 1 \\
\text{converges absolutely} \\
\text{Note that} \\
R = 1
\end{array}
\]
Examples – Power Series – 2

- Find R and the interval of convergence of \( \sum_{n=1}^{\infty} \frac{(x-2)^n}{n} \).

Series starts as: \( (x-2) + \frac{(x-2)^2}{2} + \frac{(x-2)^3}{3} + \ldots \).

We’ll apply ratio test — \( |a_n| = \frac{|x-2|^n}{n} \) (values are important).

\[
\left| \frac{a_{n+1}}{a_n} \right| = \frac{|x-2|^{n+1}}{n+1} \frac{n}{|x-2|^n} = |x-2| \frac{n}{n+1} = |x-2| \frac{1}{1+\frac{1}{n}}
\]

as \( n \to \infty \).

So: our series converges absolutely when \( |x-2| < 1 \)

i.e. \( 2-1 < x < 2+1 \) or \( 1 < x < 3 \).

\( \Rightarrow \text{it diverges when } |x-2| > 1 \) — i.e. \( x < 1 \) or \( x > 3 \).

Continued →
Examples – Power Series –2 concluded

What happens when \( |x-2| = 1 \) – i.e. when \( x = 1 \) or \( x = 3 \)?

These x-values have to be tested separately.

When \( x = 1 \) (sub x = 1 into series)

Series is \( \sum_{n=1}^{\infty} \frac{(1-2)^n}{n} = \sum_{n=1}^{\infty} \frac{(-1)^n}{n} \).

Alternating series – converges since \( b_n = \frac{1}{n} \) decreases with \( \lim_{n \to \infty} b_n = 0 \).

When \( x = 3 \)

Series is \( \sum_{n=1}^{\infty} \frac{(3-2)^n}{n} = \sum_{n=1}^{\infty} \frac{1}{n} \).

Divergent harmonic series.

Conclusion: \( R = 1 \)

Interval of convergence \( 1 \leq x < 3 \)

Convergence diverges

\( 1 \) \hspace{1cm} \( 2 - R - 3 \) \hspace{1cm} \( \rightarrow \)

\( \leftarrow \) \hspace{1cm} \( \rightarrow \)
Examples – Power Series –3

• Find R and the interval of convergence of \( \sum_{n=0}^{\infty} \frac{(-1)^n(x-1)^n}{3^n} \).

Series starts: \( 1 - \frac{(x-1)}{3} + \frac{(x-1)^2}{9} - \ldots \)

Apply root test: \( |a_n| = \frac{|x-1|^n}{3^n} \). \( \lim_{n \to \infty} |a_n|^{1/n} = \frac{|x-1|}{3} \).

Our series converges absolutely when \( \frac{|x-1|}{3} < 1 \). \(-2 < x < 4\)

Series diverges when \( x \leq -2 \) or \( x \geq 4 \). \( R = 3 \)

When \( x = -2 \), series is \( \sum_{n=0}^{\infty} \frac{(-1)^n(-3)^n}{3^n} = \sum_{n=0}^{\infty} \frac{1}{3^n} \). Diverges \( (n^{th} \text{ Term Test}) \)

When \( x = 4 \), series is \( \sum_{n=0}^{\infty} \frac{(-1)^n3^n}{3^n} = \sum_{n=0}^{\infty} (-1)^n \). ...ditto...

Interval of convergence: \(-2 < x < 4\)
Examples – Power Series – 4

- Find R and the interval of convergence of \( \sum_{n=0}^{\infty} \frac{(x+1)^n}{n!} \).

  Series starts: \( 1 + (x+1) + \frac{(x+1)^2}{2!} + \ldots \)  
  \( \text{Note: } 0! \) is defined to be 1.

  Apply ratio test (since we have factorials)

  \[
  \left| \frac{a_{n+1}}{a_n} \right| = \frac{|x+1|^{n+1}}{(n+1)!} \frac{n!}{|x+1|^n} = \frac{|x+1|}{(n+1)\cdot n!}
  \]

  \[
  \lim_{n \to \infty} \frac{|x+1|}{n+1} = 0 = L
  \]

  Conclusion: Since \( L = 0 \), our series converges absolutely for all values of \( x \).

  \( R = \infty \); Interval of convergence: \(-\infty < x < \infty\).