The Derivative as a Function

Section 2.9
The Derivative as a Function

Instead of calculating $f'$ at $x = a$, we can leave the variable as $x$ —

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

$f'(x)$ is called the derivative of $f$.

Note that it is a function of $x$, too.

• If $y = f(x)$, we have alternative notation:

$$f'(x) = y' = \frac{dy}{dx} = \frac{d}{dx} f(x)$$

You will need to recognize these other ways to write the derivative.
Differentiability Implies Continuity

Theorem: If \( f \) is differentiable at \( a \), then \( f \) must be continuous at \( a \).

Proof: Assume \( f'(a) \) is finite.

Then \( \lim_{x \to a} \left[ f(x) - f(a) \right] = \lim_{x \to a} \frac{f(x) - f(a)}{(x-a)} \)

\[ = \left[ \lim_{x \to a} \frac{f(x) - f(a)}{x-a} \right] \left[ \lim_{x \to a} (x-a) \right] \]

\[ = [f'(a)] [0] = 0 \]

Notice that \( \lim_{x \to a} [f(x) - f(a)] = 0 \) is the same as \( \lim_{x \to a} f(x) = f(a) \), so \( f \) must be continuous at \( a \).

- We say “differentiability implies continuity.”
Pictures

\((a \neq a)\)

\[\begin{align*}
\text{not continuous} & \quad \text{not differentiable} \\
\text{continuous but} & \quad \text{continuous but} \\
\text{not differentiable} & \quad \text{not differentiable} \\
\text{continuous but} & \quad \text{continuous and} \\
\text{not differentiable} & \quad \text{differentiable}
\end{align*}\]

- \(f'(a)\) exists means there is a (single-valued) finite slope at \(a\).
Relating Graphs of $f$ and $f'$

- Suppose we are given the graph of $f$, and are asked to sketch the graph of $f'$:

  For $x < A$ or $x > B$, $f$ decreases if slope is $< 0$, i.e. $f' < 0$

  At $A$ or $B$ slope is $0$, i.e. $f' = 0$

  For $A < x < B$, $f$ increases if slope is $> 0$, i.e. $f' > 0$