The Mean Value Theorem

Section 4.2
Mean Value Theorem and Rolle’s Theorem

Mean Value Theorem —

- Assume (a) \( f \) is continuous on the closed interval \([a,b]\) and (b) \( f \) is differentiable on the open interval \((a,b)\).

Then there is at least one number \( c \) in \((a,b)\) for which

\[
\frac{f'(c)}{b-a} = \frac{f(b) - f(a)}{b-a}
\]

Special case: Rolle’s Theorem —

- Assume (a), (b) and also that \( f(a) = f(b) = 0 \).

Then there is at least one number \( c \) in \((a,b)\) for which \( f'(c) = 0 \).

(see pictures on the next slide)
Pictures of Rolle’s and MV Theorems

Rolle’s Theorem \((f(a) = f(b) = 0\) and \(f'(c) = 0\))

Mean Value Theorem

\[
\frac{f(b) - f(a)}{b - a} = \text{slope at } c \text{ matches slope of } h
\]
Consequences of the Mean Value Theorem

- If $f'(x) = 0$ on $(a,b)$, then $f$ is constant on $(a,b)$.
  i.e. $f(x) = C$

- If $f'(x) = g'(x)$ on $(a,b)$, then there is a constant $C$ for which $f(x) = g(x) + C$ on $(a,b)$.

E.g. If $f(x) = 3x^2 + 1$,
  then $f$ has the same derivative as $g(x) = x^3 + x$.
  So: $f(x)$ must have the form
  \[ f(x) = x^3 + x + C \]
  where $C$ stands for a constant.
  (If we know $f'$, we can predict the form of $f$. )
Example – Finding $c$ for MV Theorem

(a) Show that $f(x) = \ln(x - 1)$ satisfies the hypotheses of the MV Theorem on $[2, 4]$. (b) Find all values $c$ that satisfy the conclusion of the MV Theorem.

(a) $f$ is continuous when $x-1>0$ (i.e. $x>1$). So $f$ is continuous on $[2, 4]$.

$$f'(x) = \frac{1}{x-1} \text{ exists for } x > 1.$$

So $f$ is differentiable on $(2, 4)$.

(b) Solve $\frac{f(4) - f(2)}{4 - 2} = f'(c)$ for $c$.

$$\frac{\ln3 - \ln1}{2} = \frac{1}{c - 1} \quad \frac{\ln3}{2} = \frac{1}{c - 1} \quad c - 1 = \frac{2}{\ln3}$$

Answer: $c = \frac{2}{\ln3} + 1$. 
Example – Exactly One Real Root

• Show \( x^3 + x - 1 = 0 \) has exactly one real root.

1 - Show there is a real root (use Intermediate Value Theorem).

\[ \text{let } f(x) = x^3 + x - 1. \quad f(0) = -1 \quad \text{and} \quad f(1) = +1. \]

\[ \begin{align*}
\text{(neg.)} & \quad \text{The Int. Value Theorem says } f \text{ must have a root in } (0, 1). \\
\text{(pos.)} & \end{align*} \]

2 - Show there cannot be any other real root.

\[ \text{let } a \text{ be the root from part 1. } f(a) = 0. \text{ If there is another root } b, \text{ Rolle's Theorem would tell us that there is a } c \text{ with } f'(c) = 0. \]

\[ \text{But } f'(c) = 3c^2 + 1, \text{ we see } f'(c) \geq 1. \]

So there cannot be another real root.
Example – Growth of $f$

- If $f(0) = -3$ and $f''(x) \leq 5$ for all $x$, how large can $f(2)$ possibly be?

  Use $a = 0$ and $b = 2$ in the MVT Theorem:

  \[
  \frac{f(2) - f(0)}{2 - 0} = f'(c) \quad \text{for some } c \text{ in } (0, 2)
  \]

  So

  \[
  \frac{f(2) + 3}{2} \leq 5 \quad \text{since } f'(c) \leq 5
  \]

  Solve for $f(2)$:

  \[
  f(2) + 3 \leq 10
  \]

  \[
  f(2) \leq 7 \quad \text{Answer}
  \]