The Substitution Rule

Section 5.5
Example 1 – Substitution in Indefinite Integrals

\[ \int x\sqrt{x^2+1} \, dx = \int \frac{1}{2} \sqrt{u} \, du \]

\[ u = x^2 + 1 \]
\[ du = 2x \, dx \]

(since \( x \, dx = \frac{du}{2} \))

\[ = \frac{1}{2} \cdot \frac{2}{3} u^{3/2} + C \]

\[ = \frac{1}{3} (x^2 + 1)^{3/2} + C \]

> Idea: replace \( \sqrt{x^2+1} \) by the simpler expression \( \sqrt{u} \).

> Rewrite \( \int \) in terms of \( u \) & \( du \).

> Answer in terms of original variable.

Check the answer by differentiating.
Example 2 – Substitution in Indefinite Integrals

\[
\int \frac{e^{\sqrt{x}}}{\sqrt{x}} \, dx = \int 2e^u \, du
\]

\[u = \sqrt{x}\]
\[du = \frac{1}{2\sqrt{x}} \, dx\]

\[
(\text{so } \frac{dx}{\sqrt{x}} = 2 \, du)
\]

\[= 2e^u + C\]
\[= 2e^{\sqrt{x}} + C\]

Idea: replace \(e^{\sqrt{x}}\) by the simpler expression \(e^u\).

Check answer by differentiating.
Useful Comments on Substitution

- You want to choose a substitution $u = \ ?$ so that the entire integral can be rewritten in terms of $u$ and $du$.

- You want the $u$-integral to be easier than the original integral, so $u = x$ never results in easier $\int$.

- Answer in terms of the original variable.
Examples 3 & 4 – Substitution in Indefinite Integrals

• \[ \int \frac{\sin x}{1+\cos^2 x} \, dx = \int -\frac{1}{1+u^2} \, du = -\tan^{-1} u + C \]
  
  \[ u = \cos x \]
  
  \[ du = -\sin x \, dx \]

• \[ \int \frac{dx}{x \ln x} = \int \frac{du}{u} = \ln |u| + C \]
  
  \[ u = \ln x \]
  
  \[ du = \frac{dy}{x} \]
Example 1 – Substitution in Definite Integrals

\[ \int_{0}^{5} \frac{dx}{\sqrt{3x+1}} = \int_{0}^{2} \frac{1}{3} u^{-1/2} \, du \]

\[ u = 3x + 1 \]
\[ du = 3 \, dx \]

\[ \text{or originally } x \text{ went from 0 to 5} \]
\[ \text{new limits must give } u \text{-range} \]
\[ \text{when } x = 0, \ u = 3(0)+1 = 1 \]
\[ \text{when } x = 5, \ u = 3(5)+1 = 16 \]

\[ = \int_{1}^{16} \frac{1}{3} u^{-1/2} \, du \]
\[ = \left[ \frac{1}{3} \cdot 2 u^{1/2} \right]_{1}^{16} \]
\[ = \frac{2}{3} (16) - \frac{2}{3} (1) = 2 \]
Example 2 – Substitution in Definite Integrals

\[
\int_{0}^{\sqrt{\pi}} x \cos(x^2) \, dx = \int_{0}^{\pi} \frac{1}{2} \cos u \, du
\]

\[u = x^2\]
\[du = 2x \, dx\]

when \( x = 0, \ u = 0 \)

\( x = \sqrt{\pi}, \ u = \pi \)

\[= \left[ \frac{1}{2} \sin u \right]^{\pi}_{0}\]

\[= \frac{1}{2} (0) - \frac{1}{2} (0) = 0\]
Example 3 – Substitution in Definite Integrals

\[ \int_{1}^{2} x \sqrt{x-1} \, dx \]

Let \( u = x - 1 \), so \( x = u + 1 \)

\[ du = dx \]

When \( x = 1 \), \( u = 0 \)

When \( x = 2 \), \( u = 1 \)

\[ = \int_{0}^{1} (u + 1) \sqrt{u} \, du \]

\[ = \int_{0}^{1} (u^{3/2} + u^{1/2}) \, du \]

\[ = \left[ \frac{2}{5} u^{5/2} + \frac{2}{3} u^{3/2} \right]_{0}^{1} \]

\[ = \left( \frac{2}{5} + \frac{2}{3} \right) - 0 = \frac{16}{15} \]