Arc Length for a Parametric Curve

Assume the smooth curve $C$ is traversed exactly once as $t$ increases from $a$ to $b$.

The differential of arc length, $ds$, is:

$$ds = \sqrt{(dx)^2 + (dy)^2}$$

so

$$ds = \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2} \, dt$$

(now $ds$ will be entirely in terms of $t$)

$$\text{Length} = \int_{t=a}^{t=b} ds$$
Example – Arc Length

Find the length of the curve $x = e^{2t} \cos t$, $y = e^{2t} \sin t$ for $0 \leq t \leq \pi/2$.

$$ds = \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2} \, dt$$

$$\frac{dx}{dt} = 2e^{2t} \cos t - e^{2t} \sin t$$

$$\frac{dy}{dt} = 2e^{2t} \sin t + e^{2t} \cos t$$

$$\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 = e^{4t} \left[4\cos^2 t - 4\cos t \sin t + \sin^2 t\right] + e^{4t} \left[4\sin^2 t + 4\sin t \cos t + \cos^2 t\right] = e^{4t} \left[5\cos^2 t + 5\sin^2 t\right] = 5e^{4t}$$

$$ds = \sqrt{5} e^{2t}$$

$$\text{Length} = \sqrt{5} \int_0^{\pi/2} e^{2t} \, dt = \frac{\sqrt{5}}{2} \left[ e^{\pi/2} - 1 \right]$$
Surface Area (of Revolution)

If the smooth parametric curve C is rotated about an axis, the differential of surface area, $dS$, is

$$dS = 2\pi r\, ds$$

where we must express $r$ and $ds$ in terms of the parameter.

(a diagram helps you to set up $r$ correctly)
Example – Surface Area

- The circle \((x - b)^2 + y^2 = a^2\) is given by the parametric equations
  \[ x = b + a \cos t, \quad y = a \sin t, \quad 0 \leq t \leq 2\pi. \]

Assume \(a < b\), and find the surface area obtained by revolving around the \(y\)-axis.

\[
\begin{align*}
\mathrm{d}s &= 2\pi n \, \mathrm{d}s \\
n &= x = b + a \cos t \\
\mathrm{d}s &= \sqrt{(\frac{dx}{dt})^2 + (\frac{dy}{dt})^2} \, \mathrm{d}t \\
&= \sqrt{(-a \sin t)^2 + (a \cos t)^2} \, \mathrm{d}t \\
&= \sqrt{a^2 (\sin^2 t + \cos^2 t)} \, \mathrm{d}t = a \, \mathrm{d}t \\
\end{align*}
\]

\[
S_{\text{surf.}} = 2\pi a \int_0^{2\pi} (b + a \cos t) \, \mathrm{d}t = 4\pi^2 ab
\]