Integrating Rational Functions – 1

- A rational function is a ratio of polynomials \( \frac{P(x)}{Q(x)} \).

- To integrate a rational function, first check whether degree \( P < \) degree \( Q \). If not, use long division to get

  \[
  \frac{P(x)}{Q(x)} = \frac{P(x)}{Q(x)} = \frac{P(x)}{Q(x)} + \frac{R(x)}{Q(x)}
  \]

where \( p \) is a polynomial of degree \( R < \) degree \( Q \).

- e.g. \( \int \frac{x^3}{x-1} \, dx = \int (x^2 + x + 1 + \frac{1}{x-1}) \, dx = \ldots \)

  (you finish this)

- For the rest of this lecture we’ll assume that we have

  \[ \int \frac{P(x)}{Q(x)} \, dx \] with degree \( P < \) degree \( Q \).
Integrating Rational Functions – 2

- **2.** Next see if $\frac{P}{Q}$ is a known derivative or if a simple substitution will help.

- **3.** Otherwise, you can use **partial fractions** to rewrite $\frac{P}{Q}$ as a sum of simpler functions.
Partial Fraction Expansions – 1

• Completely factor $Q(x)$ into linear & irreducible quadratic factors. Use only real #’s.

• If you have $n$ factors, you will rewrite $\frac{P}{Q}$ as a sum of $n$ simpler functions.

• e.g. $\frac{1}{x^3+2x} = \frac{1}{x(x^2+2)} = \frac{A}{x} + \frac{Bx+C}{x^2+2}$

  where $A, B \& C$ are uniquely determined constants.

• The numerator is constant if the denominator is based on a linear factor.

  e.g. $\frac{A}{x}$

• The numerator is linear if the denominator is based on a quadratic factor.

  e.g. $(Bx+C)/(x^2+2)$
Partial Fraction Expansions – 2

• e.g. \( \frac{1}{(x+1)(3x+2)} = \frac{A}{x+1} + \frac{B}{3x+2} \)

• e.g. \( \frac{x}{(x+2)^3} = \frac{A}{x+2} + \frac{B}{(x+2)^2} + \frac{C}{(x+2)^3} \)  
  \( \underline{3 \text{ factors}} \)  
  \( \Rightarrow 3 \text{ Terms} \)

• e.g. \( \frac{x^2+4}{(x^2+1)^2} = \frac{Ax+B}{x^2+1} + \frac{Cx+D}{(x^2+1)^2} \)  
  \( \underline{2 \text{ factors}} \)  
  \( \Rightarrow 2 \text{ Terms} \)

Next we’ll learn how to find \( A, B, C, \ldots \)
Partial Fractions Example 1

\[
\int \frac{dx}{x^2 + 3x + 2} = \int \frac{dy}{(x+1)(x+2)}
\]

\[
\frac{1}{(x+1)(x+2)} = \frac{A}{x+1} + \frac{B}{x+2}
\]

\[
1 = A(x+2) + B(x+1) \quad \text{for all } x
\]

\[
1 = (A+B)x + (2A+B)
\]

\(\text{polynom = polynom implies equal coefs. of corresponding } x\text{-powers}\)

\[
A + B = 0 \quad \text{solve: } A = 1 \quad B = -1
\]

\[
2A + B = 1
\]

Answer: \[
\int \left( \frac{1}{x+1} - \frac{1}{x+2} \right) dx = \ln | \frac{x+1}{x+2} | + C
\]
Partial Fractions Example 2

\[ \int \frac{x + 1}{x(x^2 + 1)} \, dx \]

\[ \frac{x + 1}{x(x^2 + 1)} = \frac{A}{x} + \frac{Bx + C}{x^2 + 1} \]

\[ x + 1 = A(x^2 + 1) + (Bx + C)x \]

\[ x + 1 = (A + B)x^2 + Cx + A \]

\[
\begin{align*}
A + B &= 0 \\
C &= 1 \\
A &= 1
\end{align*}
\]

\[ \begin{align*}
A &= 1, \quad B = -1, \quad C = 1
\end{align*} \]

\[ \text{Answer: } \int \left( \frac{1}{x} + \frac{-(x+1)}{x^2+1} \right) \, dx \]

\[ = \int \left( \frac{1}{x} - \frac{x}{x^2+1} + \frac{1}{x^2+1} \right) \, dx = \ln |x| - \frac{1}{2} \ln (x^2+1) + \tan^{-1} x + C \]
Partial Fractions Example 3

- A quick way to get the constants when you have distinct linear factors of \( Q(x) \) :

\[
\frac{2x+3}{x(x+1)(x-2)} = \frac{A}{x} + \frac{B}{x+1} + \frac{C}{x-2}
\]

\[2x+3 = A(x+1)(x-2) + Bx(x-2) + Cx(x+1)\]

zeros of \( Q \) are 0, -1, 2

Sub in \( x=0 \): \[3 = A(1)(-2) + 0 + 0, \quad \text{so} \quad A = -3/2\]

" \( x=-1 \): \[1 = 0 + B(-1)(-3) + 0, \quad \text{so} \quad B = 1/3\]

" \( x=2 \): \[7 = 0 + 0 + C(2)(3), \quad \text{so} \quad C = 7/6\]

\[
\int \frac{2x+3}{x(x+1)(x-2)} \, dx = -\frac{3}{2} \ln |x| + \frac{1}{3} \ln |x+1| + \frac{7}{6} \ln |x-2| + C
\]