Natural (Exponential) Growth

Natural growth problems occur when a population grows (or decays) at a rate proportional to the size \((P)\) of the population:

\[
\frac{dP}{dt} = kP
\]

\((t \text{ is Time, } k \text{ is constant})\)

\(\frac{dP}{dt}\) is the growth rate

\(k\) is the relative growth rate:

\[
k = \frac{dP/dt}{P}
\]

Applications:
- bacterial growth
- radioactive decay
- chemical concentration during reaction
- continuous compounding of interest

\(k > 0\) : growth
\(k < 0\) : decay
Solution of Natural Growth Differential Equation

- \( \{ \text{assume } \frac{dy}{dt} = ky \text{ and } y(0) = y_0 \} \quad (k \text{ constant}) \)
  \[
  \frac{dy}{y} = k\,dt \\
  \int \frac{dy}{y} = k \int dt \\
  \ln |y| = kt + C \\
  y = \pm e^{kt+C} = \pm e^C e^{kt}
  \]

  Sub in \( t=0 \) to find \( \pm e^C \):
  \[
  y(0) = y_0 = \pm e^C \quad (1)
  \]

- Solution \( \{ y(t) = y_0 e^{kt} \} \)

(This says a population with constant relative growth rate \( k \) grows exponentially)
Example – Bacteria Growth

A bacteria culture starts with 200 bacteria and grows at a constant relative rate. After 5 hours there are 7,000 bacteria.

(a) Find the relative growth rate
(b) When will the population reach 18,000?

We know \( P = P_0 e^{kt} \)

\[ P = 200 e^{kt} \]

\[ 7000 = 200 e^{5k} \]

(given \( P_0 = 200 \))

(given \( P = 7000 \) when \( t = 5 \))

Solve for \( k \):

\[ e^{5k} = \frac{7000}{200} = 35 \]

\[ 5k = \ln 35 \]

Answer (a) \( k = \frac{\ln 35}{5} \approx 0.71107 \)

For (b), we want \( t \) when \( 18000 = 200 e^{(t \ln 35)/5} \)

Answer (b) \( t = \frac{5 \ln 90}{\ln 35} \approx 6.328 \) hours.
Example – Radioactive Decay

The half-life of a radioactive sample is the time required for half the sample to decay.

- The half-life of radium-226 is 1590 years.
  (a) If a sample has mass $m$ of 250 mg, find the mass remaining after 10 years.
  (b) When will the mass be reduced to 100 mg?

we know \[ m = m_0 e^{kt} = 250 e^{kt} \]

and \[ \frac{250}{2} = 250 e^{1590k} \] (half mass remains when $t = 1590$)

we can solve this for $k$:

\[ 1590k = \ln(1/2) = -\ln2 \]

\[ k = -\frac{\ln2}{1590} \approx -0.000436 \]
Example – Radioactive Decay (concluded)

From previous slide: \( m = 250 e^{kt} \)

\( k = \frac{-\ln 2}{1590} \)

(a) Mass remaining after 10 years:

\( m = 250 e^{-(10\ln 2)/1590} \approx 239.336 \text{ mg} \)

(b) When is \( m = 100 \text{ mg} \)?

We must solve for \( t \):

\[
100 = 250 e^{-\frac{(t \ln 2)}{1590}}
\]

\[
\ln \left(\frac{10}{25}\right) = -\frac{(t \ln 2)}{1590}
\]

\[
t = -\frac{1590 \ln \left(\frac{10}{25}\right)}{\ln 2}
\]

\( \approx 2,101.866 \text{ years} \)

* Make sure the answers to (a) & (b) are reasonable.